Bag model of the quark matter (infinite extension) 2018/34.

Quarks restricted to the interior of the bag where they move as free particle → at temperature T Fermi-Dirac gas

quark matter → bag of infinite size $n_{FD} = \frac{1}{1+e^{(E_q(k)-\mu_q)/k_BT}}$

$E_q(k) = (k^2 + m_q^2)^{1/2}$ μ_q quark chemical potential $= \frac{1}{3}\mu_B$

at finite T also antiquarks are present with $\mu_{\bar{q}} = -\mu_q$

Energy density

$$\epsilon = B + \sum_q N_c \frac{2J_q+1}{2\pi^2} \int_0^\infty k^2 dk \Big[\underbrace{n(k,\mu_q)}_{\text{quarks}} + \underbrace{n(k,-\mu_q)}_{\text{antiquarks}}\Big] E_q(k)$$

↑ Bag constant → acts as potential energy density

Pressure

$$p = -B + \frac{1}{3}\sum_q N_c \frac{2J_q+1}{2\pi^2} \int_0^\infty k^2 dk \; k \underbrace{\frac{\partial E_q(k)}{\partial k}}_{\downarrow \; \frac{k}{E_q(k)}} \left[n(k,\mu_q) + n(k,-\mu_q)\right]$$

For nearly massless quarks (u,d)

$$\epsilon - 3p = 4B + \sum_q N_c \frac{2J_q+1}{2\pi^2} \int_0^\infty k^2 dk \left[n(k,\mu_q) + n(k,-\mu_q)\right] \left(E_q(k) - \frac{k^2}{E_q(k)}\right)$$

$$\approx 4B$$

Baryon number density from quarks

$$\rho_B = \frac{1}{3}\sum_q \frac{N_c(2J_B+1)}{2\pi^2} \int_0^\infty k^2 dk \left[n_q(k,\mu_q) - n(k,-\mu_q)\right]$$

Chemical potentials $\mu_u = \mu_c = \frac{1}{3}\mu_B - \frac{2}{3}\mu_q$ $\mu_d = \mu_s = \frac{1}{3}\mu_B + \frac{1}{3}\mu_q$

Mixture of u, d, s with equal density is electrically neutral

Possibility for <u>strange</u> matter 2018/35.

Edge of a neutron star $P_{edge} = 0$

$T=0$ (only quarks are excited) $\epsilon_{edge} \simeq 4B$

From hadron spectroscopy $B^{1/4} = 154,5\ MeV$

$\downarrow$

$\epsilon_{edge} \sim 5 \cdot 10^{14}\ g/cm^3 \gg$!!

The most stable nuclei $\cancel{B^{1/4}M}$ $Fe^{56} \rightsquigarrow \epsilon = 7,8\ g/cm^3$

$\left.\epsilon/\varrho\right|_{Fe} \sim 931\ MeV$

Bag model for the quark matter at the edge

$$p(T=0) = \sum_q \frac{\mu_q^4}{4\pi^2} - B = 0 \longleftrightarrow \mu_u = \mu_d = \mu_s \equiv \mu_q = \left(\frac{4\pi^2 B}{3}\right)^{1/4}$$

$$\varrho = \sum_q \frac{\mu_q^3}{3\pi^2} = \frac{\mu_q^3}{\pi^2} = 0,33\,(fm)^{-3}$$

Mind kisebb annál
erősebben
kötött!!

$$\frac{\epsilon}{\varrho} = \frac{4B\pi^2}{\mu_q^3} = \frac{4B\pi^2}{\left(\frac{4\pi^2 B}{3}\right)^{3/4}} \simeq 880\ MeV < \left.\frac{\epsilon}{\varrho}\right|_{Fe}$$

Neutrality condition $0 = \sum_q q_q \frac{\mu_q^3}{3\pi^2}$ $\mu_u = \mu_d = \mu_s$ satisfies it automatically

Possible true ground state of matter is strange? ($S \neq 0$)

The hybrid phase

Mixture of hadronic and quark matter

Phase equilibrium

$$P_{hadron}(\mu_n, \mu_e, \mathcal{R}_0, R_{03}, \cancel{\Sigma}) = P_q(\mu_n, \mu_e, B)$$

Baryonic charge density

$$\rho_{baryon} = \chi\, \rho_q(\mu_e, \mu_n) \cdot \frac{1}{3} + (1-\chi)\, \rho_H(\mu_n, \mu_e, R_{03}, \mathcal{R}_0, \Sigma)$$

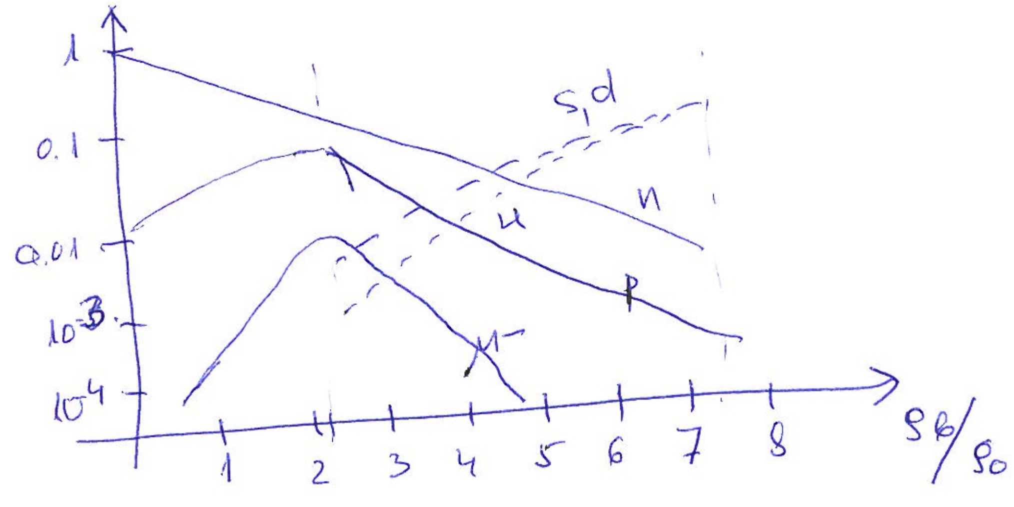
Neutrality condition

$$0 = \chi Q_q + (1-\chi) Q_{em}$$

From the 3 equations μ_n, μ_e, χ can be determined

Abundance of particle i $\quad \rho_i / \rho_b$

ρ_b is measured in proportion of ρ_0 $\quad \rho_b / \rho_0$



Equation of state and star structure 2018/37

Oppenheimer-Volkoff equation ← Einstein-equations

$$G^{\mu}_{\ \nu} = 8\pi G_N T^{\mu}_{\ \nu}$$

$$g_{\mu\nu} - \Gamma^{\gamma\beta}_{\mu\alpha} - R_{\mu\nu}$$

$$\begin{pmatrix} \epsilon & & & \\ & -p & & \\ & & -p & \\ & & & -p \end{pmatrix}$$

$$(ds)^2 = (dt)^2 e^{2\nu(r)} - (dr)^2 e^{2\lambda(r)} - r^2\left((d\varphi)^2 + \sin^2\theta (d\theta)^2\right)$$

$$\begin{pmatrix} e^{2\nu} & & & \\ & -e^{2\lambda(r)} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$$ spherical coordinates

Components of Einstein's equation

$$r^2 G^0_{\ 0} = \frac{d}{dr}\left(r\left(1 - e^{-2\lambda}\right)\right) = 8\pi G_N r^2 \epsilon(r)$$

$$G^1_{\ 1} = e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2\nu'}{r}\right) - \frac{1}{r^2} = -8\pi G_N p(r)$$

$$G^2_{\ 2} = G^3_{\ 3} = e^{-2\lambda}\left(\nu'' + \nu'^2 - \lambda'\nu' + \frac{\nu' - \lambda'}{r}\right) = -8\pi G_N p(r)$$

Solution in empty space (Schwarzschild)

$$\epsilon = p = 0$$

$$-G^0_{\ 0} = e^{-2\lambda}\left(\frac{1}{r^2} - \frac{2\lambda'}{r}\right) - \frac{1}{r^2} = 0 \qquad G^1_{\ 1} = 0 = e^{-2\lambda}\left(\frac{1}{r^2} + \frac{2\nu'}{r}\right) - \frac{1}{r^2}$$

Comparing the equations $\lambda' = -\nu'$

$\lambda + \nu = \text{const}$ but 0 since its behavior approaches the Minkowski-metric for $r \to \infty$

$y = e^{-2\lambda}$ new variable

$y' = -2\lambda' y$ $\qquad G^0_{\ 0} \Rightarrow e^{-2\lambda}(1 - 2r\lambda') = 1$ or $y + ry' = 1$ $\quad y = 1 + c\frac{1}{r}$

Convenient notation $\boxed{y = 1 - \frac{2G_N M}{r} = e^{-2\lambda}}$

$-2G_N M$ chosen as constant of integration

Finding the $g_{\mu\nu}$ elements

$$g_{11} = -\frac{1}{1-\frac{2GM}{r}} \qquad g_{00} = 1 - \frac{2G_N M}{r} \qquad r_s = 2G_N M$$

Schwarzschild singularity $r > r_s$ normal star
$r < r_s$ black hole

Interior solution

G^0_0 integrated equation $\quad r\left(1-e^{-2\lambda}\right) = 8\pi G_N \int_0^{r} dr'' \varepsilon(r'') r''^2 dr'$

$$= 2G_N M(r) > 0$$

$$e^{-2\lambda(r)} = 1 - \frac{2G_N M(r)}{r}$$

Continuous joining on the surface

General strategy for the solution

$G^0_0 \to \lambda'(r)$ is expressed

$G^1_1 \to \nu'(r)$ — " — $\nu''(r)$ with further differentiatio

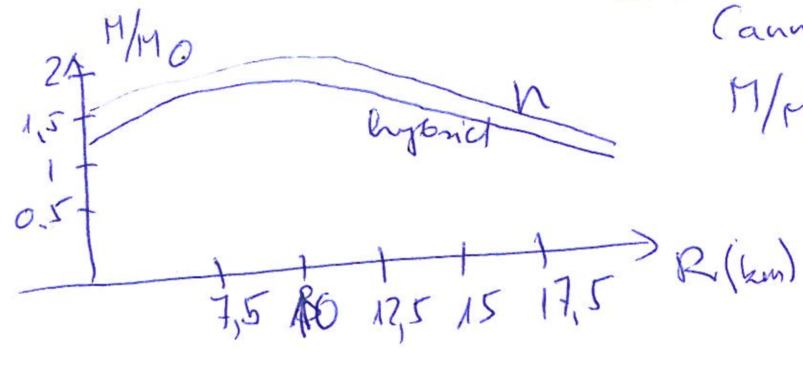
$G^2_2 \to$ ~~black remnant~~ $\quad G^2_2$ can be expressed with help of the auxiliary fields.

Its final form $\quad \dfrac{dp}{dr} = -\dfrac{(p+\varepsilon)\left(G_N M_N(r) + 4\pi r^3 p(r)\right)}{r(r-2MG_N)}$

Integration starts from the center $r=0$ $\quad \varepsilon_c \to p(\varepsilon_c)$

Endpoint of the integration $\quad p=0 \to r=R \quad M=M(R)$

$M/M_\odot - R$:



neutron-matter
Cannot explain $M/M_\odot\big|_{max} \sim 2-2,1$

Field theories constructed on the basis of approximate chiral symmetry of QCD 2018/39.

Independent kinetic terms for the $q_{R/L} = \frac{1}{2}(1 \pm \gamma_5) q = P_{R/L} q$ projections in the Dirac-action $\{\gamma_5, \gamma^\mu\}_+ = 0$

$$L^{(q)}_{kin} = \sum_{\alpha=1}^{N_f} \left[\bar{q}^\alpha_L i\gamma^\mu D_\mu q^\alpha_L + \bar{q}^\alpha_R i\gamma^\mu D_\mu q^\alpha_R \right] \qquad D_\mu = \partial_\mu - igA_\mu$$

α: flavor index $\qquad A_\mu = A^a_\mu t^a$

t^a generators of the $SU(3)$ ~~flavor~~ color

Symmetry of the kinetic theory

$q_L \to U^L q_L \quad q_R \to U^R q_R$ $\qquad U^L, U^R$ independent global transformations

$$L^{(q)}_{mass} = -\sum_{\alpha=1}^{N_f} m_\alpha \left(\bar{q}^\alpha_L q^\alpha_R + \bar{q}^\alpha_R q^\alpha_L \right)$$

violates the general symmetry only $U^L = U^R$ leads to invariance

Characteristic scale of QCD $\Lambda_{QCD} \simeq 150\,MeV$

Mass parameters $m_u, m_d \simeq 4-7\,MeV$ $\qquad SU(2)_L \otimes SU(2)_R$ is a very good approximate symmetry

$m_s \approx 80-100\,MeV$ $\qquad N_f = 3$ is acceptable but strong effects of explicit symmetry breaking

The concept is not useful for $N_f > 3$

Parametrization of the symmetry transformations

$$U_L = \exp\left\{ i\sum_a \Theta^a_L \frac{\lambda^a}{2} \right\} \qquad U_R = \exp\left\{ i\sum_a \Theta^a_R \frac{\lambda^a}{2} \right\}$$

λ^a $a = 1,2,\dots,8$ Gell-Mann matrices $\lambda^0 = \sqrt{\frac{2}{3}}\,\mathbb{1} \to$ normalisation

$$\left[\frac{1}{2}\lambda^a, \frac{1}{2}\lambda^b\right] = i f^{abc}\frac{1}{2}\lambda^c \qquad \mathrm{tr}(\lambda^a)^2 = 2 \leftarrow \text{normalisation of Gell-Mann matrices}$$

Infinitesimal transformations of the quark field $q^\alpha(x)$

2018/40.

$$q^\alpha \longrightarrow \left(\frac{1-\gamma_5}{2}\left(\delta_{\alpha\beta} + i\Theta_L^a t^a_{\alpha\beta}\right) + \frac{1+\gamma_5}{2}\left(\delta_{\alpha\beta} + i\Theta_R^a t^a_{\alpha\beta}\right)\right) q_\beta$$

$$= \left[\delta_{\alpha\beta} + i(\Theta_L^a + \Theta_R^a) t^a_{\alpha\beta} + i\gamma_5 \frac{1}{2}(\Theta_R^a - \Theta_L^a) t^a_{\alpha\beta}\right] q_\beta$$

$$\approx \left(e^{i(\Theta_V^a + \gamma_5 \Theta_R^a) t^a} q\right)_\alpha$$

L-R symmetry can be parametrized as vector axialvector symmetry $U_L(N_f) \otimes U_R(N_f) \simeq U_V(N_f) \otimes U_A(N_f)$

Group theoretical structure of the mass-breaking

$$M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix} \qquad \text{mass matrix of "light quarks"}$$

$$L_{Mass} = \bar{q} M q$$

$$= \frac{\lambda^0}{\sqrt{6}}(m_u + m_d + m_s) + \frac{\lambda^8}{\sqrt{3}}\left(\frac{1}{2}(m_u + m_d) - m_s\right) + \frac{m_u - m_d}{2}\lambda^3$$

Transformation properties of $L^{(q)}_{mass}$

$$\delta L^{(q)}_{mass} = -\bar{q}\left(1 - i\Theta_V^a t^a + i\gamma_5 \Theta_A^a t^a\right) M \left(1 + i\Theta_V t^b + i\gamma_5 \Theta_A^b t^a\right) q + \bar{q} M q$$

$$= i\bar{q}\left[\Theta_V^a [t^a, M]_- - \Theta_A^a \gamma_5 \{t^a, M\}_+\right] q$$

If $M \sim \mathbb{1}$ axial transformation remains to violate the symmetry $SU_V(3) \otimes U_V(1)$ is a symmetry

$U_A(1)$ $M=0$ classically symmetry quantum fluctuations violate this symmetry

First realisation of the nucleon model based on chiral symmetry 2018/41.

Y. Nambu – G. Jona-Lasinio $N = \begin{pmatrix} p \\ n \end{pmatrix}$

$$L_{NJL} = \bar{N}^{\alpha} i\not{\partial} N^{\alpha} - g(\bar{N}_L^{\alpha} N_R^{\beta})(\bar{N}_R^{\beta} N_L^{\alpha})$$

chiral symmetry does not allow $-m_N \bar{N} N$

Invariance of the 4-fermion term

$$\left.\begin{array}{l} \bar{N}_L^{\alpha} N_R^{\beta} \to \bar{N}_L^{\gamma} U_L^{+\gamma\alpha} U_R^{\beta\delta} N_R^{\delta} \\ \bar{N}_R^{\beta} N_L^{\alpha} \to \bar{N}_R^{\kappa} U_R^{+\kappa\beta} U_L^{\alpha\lambda} N_L^{\lambda} \end{array}\right\} (\bar{N}_L^{\gamma} N_R^{\delta})(\bar{N}_R^{\kappa} N_L^{\lambda}) \cdot \underbrace{U_R^{+\kappa\beta} U_R^{\beta\delta} \, U_L^{+\gamma\alpha} U_L^{\alpha\lambda}}_{\delta^{\kappa\delta}\delta^{\gamma\lambda}}$$

Assumption on the quantum ground state

$$\langle 0 | \bar{N}_L^{\alpha} N_R^{\beta} | 0 \rangle = \sigma_0 \delta^{\alpha\beta}$$

left hand can be nonzero only if $|0\rangle$ is not invariant under $U_L \times U_R$ $\quad U_L|0\rangle \neq |0\rangle$, $\quad U_R|0\rangle \neq |0\rangle$

Separation of the vacuum expectation

$$\bar{N}_L^{\alpha} N_R^{\beta} \to \bar{n}_L^{\alpha} n_R^{\beta} + \sigma_0 \delta^{\alpha\beta} \qquad \langle 0 | \bar{n}_L^{\alpha} n_R^{\beta} | 0 \rangle = 0$$

$$L_{NJL} = \bar{n}^{\alpha} i\not{\partial} n^{\alpha} - g\Big[(\bar{n}_L^{\alpha} n_R^{\beta})(\bar{n}_R^{\beta} n_L^{\alpha}) + \sigma_0 \underbrace{(\bar{n}_L^{\alpha} n_R^{\alpha} + \bar{n}_R^{\alpha} n_L^{\alpha})}_{\bar{n}n} + \sigma_0^2 N_f\Big]$$

$m_N = g\sigma_0$ is generated by the symmetry breaking

Generalized model closer to the known particle spectrum

$$L_{NJL} = \bar{N}^{\alpha} i\not{\partial} N^{\alpha} - G\Big[(\bar{N}^{\alpha} N^{\alpha})^2 - \big(\bar{N}^{\alpha} \tfrac{\tau^a_{\alpha\beta}}{2} \gamma_5 N^{\beta}\big)^2\Big]$$

Introduction of composite fields by adding

$$\Delta L = \left(\sqrt{G}\,\bar{N}N - \frac{1}{2}\sigma\right)^2 + \left(\sqrt{G}\, i\bar{N}\gamma_5 \frac{1}{2}\tau^a N - \frac{1}{2}\pi^a\right)^2$$

$$L_{full} = L_{NJL} + \Delta L$$
$$= i\bar{N}\not{\partial}N - \sqrt{G}\,\sigma\bar{N}N - i\sqrt{G}\,\pi^a\,\bar{N}\gamma_5\frac{\tau^a}{2}N$$
$$+ \frac{1}{4}\left(\sigma^2 + (\pi^a)^2\right)$$

$$\frac{\partial L_{full}}{\partial\sigma} = 0 = -\sqrt{G}\bar{N}N + \frac{1}{2}\sigma \qquad \frac{\partial L_{full}}{\partial\pi^a} = -i\sqrt{G}\,\bar{N}\gamma_5\frac{\tau^a}{2}N + \frac{1}{2}\pi^a = 0$$

Substituting the expression of σ, π^a into L_{full} gives back the nonlinear 4-fermion term

Assume symmetry breaking: $\sigma = \bar{\sigma} + V$

Instant consequence $m_N = \sqrt{G}V$

Quantum-equation determines the equation for V

$$0 = \left\langle \frac{\partial L}{\partial \sigma}\right\rangle\Big|_{\sigma=V} = \frac{1}{2}V - \sqrt{G}\,\langle \bar{N}N\rangle$$

filling the fermion levels to a maximum momentum Λ — see Johnson-Teller model

$$= \frac{1}{2}V - \sqrt{G}\int\frac{d^3k}{(2\pi)^3}\frac{\sqrt{G}V}{\sqrt{k^2+GV^2}}\,\Theta(\Lambda-k)\cdot\underbrace{2\cdot 2}_{\text{degeneracy}}$$

$\tilde{V} = \frac{V}{\Lambda^2}$ $\tilde{k} = \frac{k}{\Lambda}$ $\tilde{G} = G\Lambda^2$ scaled variables

$$0 = \underset{\substack{\uparrow\\ \text{symmetric solution}}}{\tilde{V}}\underbrace{\left(1 - \frac{4\tilde{G}}{\pi^2}\int_0^1 \tilde{k}^2 d\tilde{k}\left(\tilde{k}^2 + \tilde{G}\tilde{V}^2\right)^{-1/2}\right)}_{\text{symmetry breaking}}$$

for $\tilde{G} > \tilde{G}_{crit}$ $V \neq 0$ solution exists

Linear σ-model (M. Gell-Mann – M. Lévy) 2018/43.

Meson-field representing $\bar{q}_R^\alpha q_L^\beta \sim \phi^{\beta\alpha}$

$$\phi^{\beta\alpha} \longrightarrow U_L^{\beta\delta}\phi^{\delta\gamma}U_R^{+\gamma\alpha} = U_L \phi U_R^+$$

$$\phi^+ \to U_R \phi^+ U_L^+ \qquad U_L^+U_L = U_R U_R^+ = 1$$

Invariant $\quad \mathrm{tr}\,\phi^+\phi \to \mathrm{tr}\left(U_R\phi^+U_L^+U_L\phi U_R^+\right) = \mathrm{tr}\,\phi^+\phi$

Meson model $\quad L_\sigma = \frac{1}{4}\mathrm{tr}\,\partial_\mu\phi^+\partial^\mu\phi - \frac{\lambda}{16}\left(\mathrm{tr}\,\phi^+\phi - 2v^2\right)^2$

Parametrisation of $\phi = \sigma\cdot\mathbb{1} - i\tau^a\pi^a \quad \mathrm{tr}\,\tau^a\tau^b = 2\delta^{ab}$

$$\mathrm{tr}\,\phi^+\phi = 2(\sigma^2 + \pi^2)$$

$$L_\sigma = \frac{1}{2}\left(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\pi^a\partial^\mu\pi^a\right) - \frac{\lambda}{4}\left(\sigma^2+\pi^2-v^2\right)^2$$

Nucleon-meson interaction

$$L_N = \bar{N}\, i\partial\!\!\!/ N - g\left(\bar{N}_R\phi^+N_L + \bar{N}_L\phi N_R\right)$$

Symmetry breaking $\quad \sigma \to \bar{\sigma} + v$

$$U_N = -gv(\bar{N}_R N_L + \bar{N}_L N_R) - g\underbrace{\left(\bar{N}_R(\bar{\sigma} - i\tau^a\pi^a)N_L + \bar{N}_L(\bar{\sigma}+i\tau^a\pi^a)N_R\right)}_{\sigma\bar{N}N + i\bar{N}\tau^a\gamma_5 N\pi^a}$$

$$U_\sigma = \frac{\lambda}{4}\left((\bar{\sigma}+v)^2 + (\pi^a)^2 - v^2\right)^2 = \frac{\lambda}{4}\left(\bar{\sigma}^2 + (\pi^a)^2 + 2\bar{\sigma}v\right)^2$$

~~[illegible]~~ Mass-term – quadratic in the fields

$$\lambda\bar{\sigma}^2v^2 = \frac{1}{2}m_\sigma^2\bar{\sigma}^2 \qquad m_\sigma^2 = 2\lambda v^2$$

$$0\cdot(\pi^a)^2 = \frac{1}{2}m_\pi^2(\pi^a)^2 \qquad m_\pi^2 = 0 \quad \text{Goldstone fields}$$

$O(4) \to O(3)$ unbroken symmetry
with 3 generators $\to$ 3 massless fields

Non-zero π-mass due to explicit symmetry breaking

$\Delta L = -H\sigma$ (external "magnetic" field in σ-direction)

2018/44.

$$\left.\frac{\partial U(\sigma,\vec{\pi})}{\partial \sigma}\right|_{\substack{\sigma=\sigma_0 \\ \vec{\pi}=0}} = \lambda\sigma_0(\sigma_0^2 - v^2) - H = 0 \qquad \sigma_0^2 - v^2 = \frac{H}{\lambda\sigma_0} \approx \frac{H}{\lambda v}$$

The term proportional to $(\pi^a)^2$ in the potential

$$\left.\frac{\lambda}{4} 2(\pi^a)^2\left((\sigma_0+\bar{\sigma})^2 - v^2\right)\right|_{\bar{\sigma}=0} = \frac{\lambda}{2}(\pi^a)^2 \cdot \frac{H}{\lambda v} \to \boxed{m_\pi^2 = \frac{H}{v}}$$

Requirement $m_\pi^2 \ll m_\sigma^2$: $\frac{H}{v} \ll 2\lambda v^2$

$m_\pi^2 \ll m_V^2$ $\frac{H}{v} \ll g^2 v^2$

"Pure" Goldstone-theory for energy scale much below of m_σ

The fluctuations of the massive degrees of freedom are not excited

$\Phi = S\,\mathcal{U}$ $\qquad \mathcal{U} = \exp\left(i\frac{\tau^a \varphi^a}{2F}\right)$

$\langle S \rangle = v$ cannot be excited frozen to v

$\mathrm{tr}\,\phi^+\phi = S^2(x)\,\mathrm{tr}\,\mathcal{U}^+\mathcal{U} = S^2 \qquad S = \bar{S} + v$

$(\mathrm{tr}\,\phi^+\phi - 2v^2)^2 = 4(\bar{S}^2 + 2\bar{S}v)^2$ only $\bar{S}$ is massive!

φ^a even does not appear in the potential energy

$$\mathrm{tr}\,\partial_\mu\phi^+\partial^\mu\phi = \partial_\mu\bar{S}\partial^\mu\bar{S}\,\mathrm{tr}\,\mathcal{U}^+\mathcal{U}$$
$$+ (\bar{S}+v)\partial_\mu\bar{S}\,\mathrm{tr}\underbrace{(\mathcal{U}^+\partial^\mu\mathcal{U} + \mathcal{U}\partial^\mu\mathcal{U}^+)}_{\partial_\mu(\mathcal{U}^+\mathcal{U})=0} +$$
$$+ (\bar{S}+v)^2\,\mathrm{tr}\,\partial_\mu\mathcal{U}^+\partial^\mu\mathcal{U}$$

$\bar{S}$ is not excited $\quad L_\varphi = \frac{v^2}{4} \operatorname{tr} \partial_\mu \mathcal{U}^\dagger \partial^\mu \mathcal{U}$ 2018/45.

Goldstone-dynamics

$m_{Goldstone} = 0$ obvious

Generating mass via explicit symmetry breaking

$$L_{mass} = \operatorname{tr}\left[h^\dagger \phi + h \phi^\dagger\right] = 2B_0 v \operatorname{tr}\left[\mathcal{M}\left(\mathcal{U} + \mathcal{U}^\dagger\right)\right]$$

$$\mathcal{M} = \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} \quad \text{in 2-flavor case}$$

$$\operatorname{tr}\Big[\mathcal{M}\Big(1 + \frac{i\varphi^a\tau^a}{2F} - \frac{1}{2}\frac{\varphi^a\varphi^b\tau^a\tau^b}{(2F)^2} + \dots$$

$$+ 1 - \frac{i\varphi^a\tau^a}{2F} - \frac{1}{2}\frac{\varphi^a\varphi^b\tau^a\tau^b}{(2F)^2}\Big)\Big]$$

$$\rightarrow -\frac{1}{(2F)^2} \operatorname{tr}\left(\left[\frac{m_u+m_d}{2} + \frac{m_u-m_d}{2}\tau_3\right]\underbrace{\tau^a\tau^b}_{\delta^{ab} + i\varepsilon^{abc}\tau^c}\right)\varphi^a\varphi^b$$

$$= -\frac{1}{(2F)^2}(m_u+m_d)\varphi^a\varphi^a$$

$\operatorname{tr}\tau^3 = 0$

$\operatorname{tr}\tau^3\tau^c = 2\delta^{3c}$

but $\varepsilon^{abc}\varphi^a\varphi^b = 0$

$$4B_0 v \frac{1}{(2F)^2}(m_u+m_d) = m_\pi^2$$

Size of F? from requiring <u>canonical</u> kinetic term

$$\frac{1}{4}v^2 \operatorname{tr}\partial_\mu\left(1 - \frac{i\tau^a\varphi^a}{2F}\right)\partial^\mu\left(1 + \frac{i\tau^b\varphi^b}{2F}\right) = \frac{1}{4}v^2\,\partial_\mu\varphi^a\partial^\mu\varphi^b \frac{1}{(2F)^2}\underbrace{\operatorname{tr}\tau^a\tau^b}_{2\delta^{ab}}$$

$$= \frac{1}{2}\partial_\mu\varphi^a\partial^\mu\varphi^a \cdot \frac{v^2}{(2F)^2}$$

$v = 2F$ leads to the conventional normalisation of the kinetic term.

Similar steps for $N_f = 3$ flavors

Pion-pion scattering amplitude in the lowest order of chiral perturbation theory (nonlinear σ-model)

2018/46.

$$\varphi^{c}(p_1) + \varphi^{d}(p_2) \to \varphi^{e}(p_3) + \varphi^{f}(p_4)$$

Quantized φ^a-field

$$\hat{\varphi}_0^a(\underline{x},t) = \int \frac{d^3 p}{(2\pi)^3} \left(\hat{a}^a(p) e^{-ipx} + \hat{a}^{a+}(p) e^{ipx} \right)$$

$$px = p_0 x_0 - \underline{p}\underline{x} \qquad p_0 = (\underline{p}^2 + m_\pi^2)^{1/2}$$

Creation of one-particle state:

$$\hat{a}^{a+}(\underline{p})|0\rangle = |\underline{p}, a\rangle$$

Initial state $\quad a^c(\underline{p}_1)^+ a^d(\underline{p}_2)^+ |0\rangle = |\underline{p}_1 c; \underline{p}_2 d\rangle$

Intermediate evolution - absorption of these particles:

$$\hat{\varphi}^l \hat{\varphi}^j |\underline{p}_1 c; \underline{p}_2 d\rangle$$

Final state $\quad \langle \underline{p}_3 e; \underline{p}_4 f| = \langle 0| a^e(p_3) a^f(p_4)$

Intermediate evolution - absorption of these operators

$$\langle 0| a^e(p_3) a^f(p_4) \hat{\varphi}^m \hat{\varphi}^n$$

Lowest order contribution 4-th power in $\hat{\varphi}$

$$L_{NL}^{(kinetic)} = \frac{1}{4} v_0^2 \, \mathrm{tr}\, \partial_\mu e^{i\frac{\hat{\varphi}^a \tau^a}{v_0}} \, \partial^\mu e^{-i\frac{\hat{\varphi}^b \tau^b}{v_0}}$$

expansion to cubic terms contributes

$$= \frac{1}{4} v_0^2 \, \mathrm{tr}\Big[\partial_\mu \Big(i\frac{\hat{\varphi}}{v_0} - \frac{1}{2}\frac{\hat{\varphi}^2}{v_0^2} - i\frac{1}{6}\frac{\hat{\varphi}^3}{v_0^3}\Big) \partial^\mu \Big(-i\frac{\hat{\varphi}}{v_0} - \frac{1}{2}\frac{\hat{\varphi}^2}{v_0^2} + i\frac{1}{6}\frac{\hat{\varphi}^3}{v_0^3}\Big)$$

2018/47.

$$L^{kin}_{(4)} = -\frac{1}{12 V_0^2} \operatorname{tr}\left(\partial_\mu \hat{\varphi} \cdot \partial^\mu \hat{\varphi}^3\right) + \frac{1}{16 V_0^2} \operatorname{tr} \partial_\mu \hat{\varphi}^2 \partial^\mu \hat{\varphi}^2$$

Tracing: $\operatorname{tr} \tau^a \tau^b \tau^c \tau^d = \operatorname{tr}\left[(\delta^{ab} + i\varepsilon^{abq}\tau^q)(\delta^{cd} + i\varepsilon^{cdp}\tau^p)\right]$

$$= 2\delta^{ab}\delta^{cd} - 2\varepsilon^{abp}\varepsilon^{cdp}$$

$$= 2\left[\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right]$$

Explicit expressions for the two terms

First:

$$-\frac{1}{12 V_0^2} 2\left[\partial_\mu \hat{\varphi}^a \partial^\mu (\hat{\varphi}^a \hat{\varphi}^d \hat{\varphi}^d) + \cancel{\partial_\mu \hat{\varphi}^a \partial^\mu \hat{\varphi}^b \hat{\varphi}^b \hat{\varphi}^a} - \cancel{\partial_\mu \hat{\varphi}^a \partial^\mu \hat{\varphi}^b \hat{\varphi}^a \hat{\varphi}^b}\right]$$

$$= -\frac{1}{6 V_0^2} \partial_\mu \hat{\varphi}^a \partial^\mu \hat{\varphi}^a \hat{\varphi}^d \hat{\varphi}^d$$

Second $= \frac{1}{8 V_0^2} \partial_\mu (\hat{\varphi}^a \hat{\varphi}^a) \partial^\mu (\hat{\varphi}^b \hat{\varphi}^b)$

Contribution from the mass term

$$2 B V_0 \operatorname{tr}\left(\bar{m} + \tfrac{1}{2}(m_u - m_d)\tau^3\right)\left(2 - 2\tfrac{1}{2}\frac{\varphi^2}{V_0^2} + \underline{\frac{2}{V_0^4}\frac{\varphi^4}{24}}\right)$$

$\bar{m} = \frac{1}{2}(m_u + m_d)$

$\Delta m = \frac{1}{2}(m_u - m_d)$

$$\rightarrow \frac{1}{3 \cdot 2} B_0 \frac{1}{V_0^3} 2 \operatorname{tr}\left[\bar{m}\, \hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^c \hat{\varphi}^d + \Delta m\, \hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^c \hat{\varphi}^d \cdot \operatorname{tr} \tau^3 \tau^a \tau^b \tau^c \tau^d\right]$$

$$\operatorname{tr} \tau^3 (\delta^{ab} + i\varepsilon^{abq}\tau^q)(\delta^{cd} + i\varepsilon^{cdp}\tau^p) = 2i\varepsilon^{abq}\delta^{q3}\delta^{cd} + 2i\varepsilon^{cdp}\delta^{ab} - \varepsilon^{abq}\varepsilon^{cdp} i \varepsilon^{qp3}$$

by the symmetry of $\hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^c \hat{\varphi}^d$ this contribution is zero

$$B_0 \frac{1}{3} \bar{m} \frac{1}{V_0^3} \hat{\varphi}^a \hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^b$$

The interaction Hamiltonian density 2018/48.

$$H_I = -L_I \qquad \text{mass selection} \qquad m_\pi^2 = \frac{8 B_0 \bar{m}}{V_0}$$

$$H_I = \frac{1}{6V_0^2} \partial_\mu \hat{\varphi}^a \partial^\mu (\hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^b) - \frac{1}{8V_0^2} \partial_\mu (\hat{\varphi}^a \hat{\varphi}^a) \partial^\mu (\hat{\varphi}^b \hat{\varphi}^b)$$

$$- \frac{m_\pi^2}{24 V_0^2} (\hat{\varphi}^a \hat{\varphi}^a)(\hat{\varphi}^b \hat{\varphi}^b)$$

Transition matrix element:

$$S_{fi}^{(1)} = \int d^4x \, \langle \text{final state} | H_I | \text{initial state} \rangle$$

The contributions come from all possible pairwise association of a certain $\hat{\varphi}^c$ operator with the possible creation and annihilation operators

$$\langle 0 | a^e(p_3) a^f(p_4) \partial_\mu \hat{\varphi}^a \partial^\mu (\hat{\varphi}^a \hat{\varphi}^b \hat{\varphi}^b) \, a^+(p_1)^c a^+(p_2)^d | 0 \rangle$$

2 choices

2x choices

x - dependents factors

$$e^{ip_4 x} \quad ip_4 \quad i(p_3 - p_1 - p_2) \, e^{ip_3 x - ip_1 x - ip_2 x}$$

x integration $(2\pi)^4 \delta(p_4 + p_3 - p_1 - p_2)$

Index combination $\delta^{af} \delta^{ae} \cdot \delta^{bc} \delta^{bd} = \delta^{ef} \delta^{cd}$

other pairings $\times 2$

Contribution to the S-matrix element

$$ip_4 \, i(p_3 - p_1 - p_2) \delta^{ef} \delta^{cd} (2\pi)^4 \delta(p_4 + p_3 - p_1 - p_2) = m_\pi^2 \delta^{ef} \delta^{cd} (2\pi)^4 \delta(p_4 + p_3 - p_1 - p_2)$$

Further pairings 2018/49

$$\langle 0| a^e(p_3)\, a^f(p_4)\, \partial_\mu \varphi^a\, \partial^\mu(\varphi^a \varphi^b \varphi^b)\, a^{c+}(p_1)\, a^{d+}(p_2)|0\rangle$$

$$\langle 0| a^e(p_3)\, a^f(p_4)\, \partial_\mu \varphi^a\, \partial^\mu(\varphi^a \varphi^b \varphi^b)\, a^{c+}(p_1)\, a^{d+}(p_2)|0\rangle$$

Equivalent terms with $p_3 \leftrightarrow p_4$ exchange

with exchanging leftward and rightward pairings

12 terms in the contribution

$$\Rightarrow 2\,(2\pi)^4\delta(p_3+p_4-p_1-p_2)\Big[-p_4(p_3-p_1-p_2)(\delta^{ef}\delta^{cd}+\delta^{ed}\delta^{cf}+\delta^{ec}\delta^{fd})$$

$+(-1)p_3(p_4-p_1-p_2)$ (same isospin factor)

$+(-1)p_1(p_2-p_3-p_4)$ (same isospin factor)

$+(-1)p_2(p_1-p_3-p_4)$ (same isospin factor)$\Big]$

$$= 4m_\pi^2\cdot 2\,(\delta^{ef}\delta^{cd}+\delta^{cf}\delta^{ed}+\delta^{fd}\delta^{ce})$$

$$\Rightarrow \frac{8m_\pi^2}{6v_0^2}(\delta^{ef}\delta^{cd}+\delta^{cf}\delta^{ed}+\delta^{fd}\delta^{ce})$$

Similar steps for the other two terms

$$4\cdot 2\Big[i(p_3+p_4)\,i(p_1+p_2)\,\delta^{ef}\delta^{cd}+\cancel{i(p_4-p_1)\,i(p_3-p_2)\,\delta^{cd}\delta^{eb}}$$
$$+\,i(p_4-p_1)\,i(p_3-p_2)\,\delta^{cf}\delta^{ed},\ i(p_4-p_2)\,i(p_3-p_1)\,\delta^{ec}\delta^{fd}\Big]$$

$$\Rightarrow -\cancel{8}\frac{1}{8v_0^2}\Big[s\,\delta^{ef}\delta^{cd}+u\,\delta^{cf}\delta^{ed}+t\,\delta^{cd}\delta^{fd}\Big]$$

Tömegtag $-\frac{M_\pi^2}{24v_0^2}\,8(\delta^{ef}\delta^{cd}+\delta^{cf}\delta^{ed}+\delta^{ec}\delta^{fd})$

$s=(p_1+p_2)^2$
$t=(p_3-p_1)^2$
$u=(p_4-p_1)^2$

Putting together the 3 terms $S_{fi} = (2\pi)^4 i\, \delta(p_1+p_2-p_3-p_4)$

2018/50.

$$T_{fi} = -\frac{1}{v^2}\Big[(s-m_\pi^2)\delta^{ef}\delta^{cd} + (t-m_\pi^2)\delta^{ec}\delta^{fd} + (u-m_\pi^2)\delta^{cf}\delta^{ed} \Big]$$

$$T_{fi}(s=t=u=m_\pi^2)=0 \quad \text{(Adler-zero)}$$

Better approximation – higher derivatives in the Lagrangian (preserving $SU(2)_V \times SU(2)_A$)