Renormalisability of 2PI-Hartree approximation in the broken symmetry phase

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Plan of the talk:

- Research interests of the Eötvös group in the field of quantum field theoretical functional methods
- The 2PI-Hartree approximation(s) to the effective action of scalar models
- Analysis of renormalisability in the broken symmetry phase
- Application to DS-equations:
  Coleman-Weinberg symmetry breaking in an extended Higgs sector

Collective phenomena of interacting quantum fields in and out-of equilibrium (since cca. 1990)

Phase transformations of strong and electroweak matter (phase diagrams, equations of state, etc.)

Higgs phenomenon in extensions of the Standard Model (symmetry breaking mechanism, real time dynamics, also in expanding Universe)

Former and present members

Antal Jakovác	Budapest Technical University (at present Humboldt Fellow in Wuppertal)
Péter Petreczky	Brookhaven National Laboratory
Zsolt Szép	Research Institute for Solid State Physics and Optics (on one-month stay in Santa Barbara)
Szabolcs Borsányi	University of Sussex
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General features & phenomenological applications
for relevant field theoretical (effective) models
consistent implementation of the (non-perturbative) renormalisation program

2PI-Hartree approximate solutions of scalar field theories

A general class:

$$L = \frac{1}{2}[\partial_\mu \sigma_a \partial^\mu \sigma_a - \mu^2 \sigma_a \sigma_a] - \frac{1}{3} F_{abcd} \sigma_a \sigma_b \sigma_c \sigma_d$$

$$\sigma_a \to \sigma_a + v_a$$

$$\begin{aligned} V[v_a, G_{ab}] = &\frac{1}{2}\mu^2 v_a v_a + \frac{1}{3} F_{abcd} v_a v_b v_c v_d \\ &- \frac{i}{2} \int_k D^{-1}_{ab}(k) G_{ba}(k) - \frac{i}{2} \mathrm{Tr} \int_k (\ln G^{-1}_{ab}(k)) \\ &+ \tilde{F}_{abcd} \int_k G_{ab}(k) \int_p G_{cd} + V_{ct}. \end{aligned}$$

$$iD^{-1}_{ab}(k) = (k^2 - \mu^2)\delta_{ab} - 4F_{abcd} v_c v_d$$

$$\begin{aligned} V_{ct} = V^{(0)}_{ct} + V^{(2)}_{ct} + V^{(4)}_{ct} & \\ V^{(0)}_{ct} &= \delta \tilde{F}_{abcd} \int_k G_{ab}(k) \int_p G_{cd} \\ V^{(2)}_{ct} &= \frac{1}{2} \delta \hat{\mu}^2 \int_k G_{aa} + 4 v_a v_b \delta \hat{F}_{abcd} \int_k G_{cd}(k) \\ V^{(4)}_{ct} &= \frac{1}{2} \delta \mu^2 v_a v_a + \frac{1}{3} \delta F_{abcd} v_a v_b v_c v_d. \end{aligned}$$

2 different F_{abcd} coupling tensors, 3 different δF_{abcd} coupling counter-tensors, 2 different $\delta\mu^2$ mass-counter terms

respecting the symmetry of the model can be introduced.

Gap equations and equation of state

Stationarity equations $$\frac{\delta V}{\delta v_a} = 0, \quad \frac{\delta V}{\delta G_{ab}(k)} = 0.$$

$$iG^{-1}_{ab}(k) = k^2 - M^2_{ab},$$

$$M^2_{ab} = m^2_{ab} + 4\tilde{F}_{abcd}\int_k G_{cd}(k) + \delta\hat{m}^2_{ab} + 4\delta\tilde{F}_{abcd}\int_k G_{cd}(k),$$

$$0 = v_b\left[\mu^2\delta_{ab} + \delta\mu^2\delta_{ab} + \frac{4}{3}(F_{abcd} + \delta F_{abcd})v_c v_d + 4(F_{abcd} + \delta\hat{F}_{abcd})\int_k G_{cd}(k)\right].$$

$$m^2_{ab} = \mu^2\delta_{ab} + 4F_{abcd}v_c v_d, \qquad \delta\hat{m}^2_{ab} = \delta\hat{\mu}^2\delta_{ab} + 4\delta\hat{F}_{abcd}v_c v_d.$$

Decomposition of $F, \tilde{F}, \delta F, \delta\tilde{F}, \delta\hat{F}$ into the linear combination of rank-4 invariant tensors t^u_{abcd}:

$$F = \sum_u \lambda_u t^u, \tilde{F} = \sum_u \tilde{\lambda}_u t^u,$$

$$\delta F = \sum_u \delta\lambda_u t^u, \delta\tilde{F} = \sum_u \delta\tilde{\lambda}_u t^u, \delta\hat{F} = \delta\hat{\lambda}_u t^u, \qquad t^u * t^v = \sum_w g_{uvw} t^w$$

Example: $O(N)$ model

Two independent $O(N)$ invariant rank-4 tensors:

$t^1_{abcd} = \delta_{ab}\delta_{cd}, \quad t^2_{abcd} = \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}$

$$F = \frac{1}{24N}\lambda(t^1 + t^2), \qquad \tilde{F} = \frac{1}{24N}\lambda(\alpha_1 t^1 + \alpha_2 t^2),$$

$$\delta F = \frac{1}{24N}\delta\lambda_4 t^1, \qquad \delta\hat{F} = \frac{1}{24N}(\delta\lambda_2^A t^1 + \delta\lambda_2^B t^2), \qquad \delta\tilde{F} = \frac{1}{24N}(\delta\lambda_0^A t^1 + \delta\lambda_0^B t^2).$$

$O(N)$ equation of state:

$$0 = v\Big[\mu^2 + \delta\mu^2 + \frac{1}{6N}[(\lambda + \delta\lambda_4)v^2 + T(M_\sigma^2)(3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B) + T(M_\pi^2)(N-1)(\lambda + \delta\lambda_2^A)]\Big].$$

$O(N)$ gap equations:

$$M_\pi^2 = \mu^2 + \delta\hat{\mu}^2 + \frac{1}{6N}(\lambda + \delta\lambda_2^A)v^2 + \frac{1}{6N}T(M_\pi^2)[2(\alpha_2\lambda + \delta\lambda_0^B) + (N-1)(\alpha_1\lambda + \delta\lambda_0^A)] + \frac{1}{6N}T(M_\sigma^2)(\alpha_1\lambda + \delta\lambda_0^A),$$

$$M_\sigma^2 = \mu^2 + \delta\hat{\mu}^2 + \frac{1}{6N}(3\lambda + 2\delta\lambda_2^B + \delta\lambda_2^A)v^2 + \frac{1}{6N}T(M_\sigma^2)((2\alpha_2 + \alpha_1)\lambda + \delta\lambda_0^A + 2\delta\lambda_0^B) + \frac{N-1}{6N}T(M_\pi^2)(\alpha_1\lambda + \delta\lambda_0^A).$$

$$T(M^2) \equiv \int_k \frac{i}{k^2 - M^2} = \frac{\Lambda^2}{16\pi^2} + M^2 T_d + T_F(M^2), \qquad T_d = -\frac{1}{16\pi^2}\ln\frac{e\Lambda^2}{M_0^2}.$$

Two special cases:

I. $F_{abcd} = \tilde{F}_{abcd} \quad \leftrightarrow \quad \alpha_1 = \alpha_2 = 1$

Conventional 2PI weighting

Compatibility of the equation of state with the σ gap equation:

$$v\left(M_\sigma^2 - \frac{\lambda}{3N}v^2\right) = 0,$$

$$\delta\mu^2 = \delta\hat{\mu}^2, \;\; \delta\lambda_0^A = \delta\lambda_2^A, \;\; \delta\lambda_4 = \delta\lambda_2^A + 2\delta\lambda_2^B$$

II. $\alpha_1 = 3, \alpha_2 = -(N-1)$

"Symmetrized" model of Ivanov, Riek, Hees and Knoll (2005)

Compatibility of the equation of state with the π gap equation:

$$M_\pi^2 = 0 \qquad \text{Goldstone theorem fullfilled}$$

$$\delta\mu^2 = \delta\hat{\mu}^2, \;\; \delta\lambda_0^A = \delta\lambda_2^A + 2\delta\lambda_2^B, \;\; \delta\lambda_2^A = \delta\lambda_0^A + \frac{2}{N-1}\delta\lambda_0^B, \;\; \delta\lambda_4 = \delta\lambda_2^A$$

Renormalisability conditions

Illustrative example: Difference of the two gap equations:

$$M_\sigma^2 - M_\pi^2 = \frac{1}{3N}\left[(\lambda + \delta\lambda_2^B)v^2 + (T(M_\sigma^2) - T(M_\pi^2))(\alpha_2\lambda + \delta\lambda_0^B)\right]$$

Separation of finite and infinite parts:

$$M_\sigma^2 - M_\pi^2 = \frac{1}{3N}\left[\lambda v^2 + \lambda(T_F(M_\sigma^2) - T_F(M_\pi^2))\alpha_2\lambda\right],$$

$$0 = \frac{1}{3N}\left[\delta\lambda_2^B v^2 + (T_F(M_\sigma^2) - T_F(M_\pi^2))\delta\lambda_0^B + (M_\sigma^2 - M_\pi^2)T_d(\alpha_2\lambda + \delta\lambda_0^B)\right].$$

Substituting $M_\sigma^2 - M_\pi^2$ from the finite equations into the renormalisation condition coefficients of $T_F(M_\sigma^2) - T_F(M_\pi^2)$ and of v^2 vanish separately:

$$\delta\lambda_2^B + \frac{1}{3N}\lambda T_d(\alpha_2\lambda + \delta\lambda_0^B) = 0, \qquad \delta\lambda_0^B + \frac{1}{3N}\alpha_2\lambda T_d(\alpha_2\lambda + \delta\lambda_0^B) = 0.$$

Renormalised equation of state and gap equations

$$0 = \mu^2 + \frac{\lambda}{6N}\left[v^2 + 3T_F(M_\sigma^2) + (N-1)T_F(M_\pi^2)\right],$$

$$M_\pi^2 = \mu^2 + \frac{\lambda}{6N}\left[v^2 + (2\alpha_2 + (N-1)\alpha_1)T_F(M_\pi^2) + \alpha_1 T_F(M_\sigma^2)\right],$$

$$M_\sigma^2 = \mu^2 + \frac{\lambda}{6N}\left[3v^2 + (N-1)\alpha_1 T_F(M_\pi^2) + (2\alpha_2 + \alpha_1)T_F(M_\sigma^2)\right].$$

Renormalisation of the equation of state

The condition

$$\begin{aligned}0 = \delta\mu^2 + \frac{1}{6N}\delta\lambda_4 v^2 + \frac{1}{6N}\left[T_F(M_\sigma^2)(\delta\lambda_2^A + 2\delta\lambda_2^B) + T_F(M_\pi^2)(N-1)\delta\lambda_2^A\right] \\ + \frac{\Lambda^2}{96N\pi^2}\left[(N+2)\lambda + N\delta\lambda_2^A + 2\delta\lambda_2^B\right], \\ + \frac{1}{6N}T_d\left[(N-1)M_\pi^2(\lambda + \delta\lambda_2^A) + M_\sigma^2(3\lambda + \delta\lambda_2^A + 2\delta\lambda_2^B)\right].\end{aligned}$$

Substitute M_π^2 and M_σ^2 from the renormalised gap equations into the last line equate separately to zero the coefficients of $T_F(M_\sigma^2)$, $T_F(M_\pi^2)$, v^2 and the constant piece
$\to$ 4 equations for the determination of $\delta\lambda_2^A, \delta\lambda_2^B, \delta\lambda_4, \delta\mu^2$.

Similar set of equations from the gap equations.

How can we be sure that they are mutually consistent?

General analysis

The resulting mass matrix is diagonalised by an orthogonal matrix O_{ci}:

$$\tilde{M}_i^2\delta_{ij} = O_{ci}M_{cd}^2 O_{dj}$$

Renormalised equation for the squared mass matrix:

$$M_{ab}^2 = \mu^2\delta_{ab} + 4F_{abcd}v_c v_d + 4\tilde{F}_{abcd}O_{ci}T_F(\tilde{M}_i^2)O_{di}$$

Divergence cancellation conditions

"Environment" (μ^2, T) dependent subdivergences:

$$4O_{el}T_F(\tilde{M}_l^2)O_{fl}[\delta\tilde{F}_{abef} + 4T_d(\tilde{F}+\delta\tilde{F})_{abcd}\tilde{F}_{cdef}] = 0,$$

The braketed expression is written as linear combination of t^u.
If the number of subspaces of degenerate squared masses is equal or larger than the number of independent rank-4 invariant tensors:

Vanishing of all elements of the braketed tensor yields the solution.

Determination of $\delta\hat{F}_{abef}$

Overall divergences $\sim v_e v_f$ piece:

$$4v_e v_f[\delta\hat{F}_{abef} + 4T_d(\tilde{F} + \delta\tilde{F})_{abcd}F_{cdef}] = 0,$$

Example: if $F = \tilde{F}$, then $v_e v_f(\delta\hat{F} - \delta\tilde{F})_{abef} = 0$.

Determination of $\delta\hat{\mu}^2$

v-independent piece:

$$\delta\hat{\mu}^2\delta_{ab} + 4(\tilde{F}_{abcd} + \delta\tilde{F}_{abcd})(\frac{\Lambda^2}{16\pi^2} + \mu^2 T_d)\delta_{cd} = 0$$

. Equation of state is compatible if

$$\left(\frac{1}{3}\delta F_{abcd} - \delta\tilde{F}_{abcd}\right) v_b v_c v_d = 0.$$

Back to the $O(N)$ model (Case I e.g. $F = \tilde{F}$)

Solution for the coefficients:

$$\delta\lambda_0^A = \frac{\lambda}{6N} T_d[(N+4)\lambda + (N+2)\delta\lambda_0^A + 2\delta\lambda_0^B],$$

$$\delta\lambda_0^B = \frac{\lambda}{3N} T_d[\lambda + \delta\lambda_0^B].$$

Parametrisation of the potential energy counterterm:

$$\delta F_{abcd} = \frac{1}{24N} \delta\lambda_4(\delta_{ab}\delta_{bc} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$$

leads to

$$\delta\lambda_4 = \delta\lambda_0^A + 2\delta\lambda_0^B$$

When $N \to \infty$

$$\delta\lambda_0^A = -\frac{\lambda^2}{6} T_d \frac{1}{1 - \frac{\lambda}{6} T_d}, \qquad \delta\lambda_0^B \sim \mathcal{O}(1/N)$$

which leads to a unique quartic counter coupling $\delta\lambda_4 = \delta\lambda_0^A$ and reproduces the exact result of the leading order large N analysis.

Remark:

The equations for $\delta\lambda_0^A, \delta\lambda_0^B$ coincide with those which can be derived with the method of iterative renormalisation (Blaizot, Iancu, Reinosa, 2004) where one looks for the self-energy in form of infinite series:

$$\Pi_{ab}(k) = \sum_n \Pi_{ab}^{(n)}(k), \qquad \delta\lambda_0^A = \sum_n \delta\lambda_0^{A(n)}, \qquad \delta\lambda_0^B = \sum_n \delta\lambda_0^{B(n)}$$

and solves the gap equations iteratively.

Coleman-Weinberg type symmetry breaking in extended Higgs-sector

Model: A single real order parameter field representing the SM Higgs sector is supplemented by a scalar N-plet (Szép & Patkós, 2006)

$$L_{Higgs} = \frac{1}{2}[\partial_\mu\sigma\partial^\mu\sigma - (m_\sigma^2 + \delta m_\sigma^2)\sigma^2] - \frac{1}{24N}(\lambda_{\sigma\sigma} + \delta\lambda_{\sigma\sigma})\sigma^4$$

$$\Delta L_{Higgs} = \frac{1}{2}[\partial_\mu\psi_i\partial^\mu\psi_i - (m_\psi^2 + \delta m_\psi^2)\psi_i^2] - \frac{1}{24N}(\lambda_{\psi\psi} + \delta\lambda_{\psi\psi})(\psi_i^2)^2 - \frac{1}{12N}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})\psi_i^2\sigma^2.$$

Dyson-Schwinger equations in the $N \to \infty$ limit in the symmetry broken phase $(\sigma \to \sqrt{N}v + \sigma)$

$$\sqrt{N}v\left[m_\sigma^2 + \delta m_\sigma^2 + \frac{1}{6}\Big((\lambda_{\sigma\sigma} + \delta\lambda_{\sigma\sigma})v^2 + (\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})T(M_\psi^2)\Big)\right] = 0,$$

$$m_\psi^2 + \delta m_\psi^2 + \frac{1}{6}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})v^2 + (\lambda_{\psi\psi} + \delta\lambda_{\psi\psi})T(M_\psi^2)) = M_\psi^2,$$

$$iG_\sigma^{-1}(p) = p^2 - m_\sigma^2 - \delta m_\sigma^2 - \frac{1}{2}(\lambda_{\sigma\sigma} + \delta\lambda_{\sigma\sigma})v^2 - \frac{1}{6}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})T(M_\psi^2) - i\sqrt{N}v\frac{1}{6}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})\int_k G_\psi(p+q-k)G_\psi(k)\Gamma_{\psi\psi\sigma}(p-k,k,-p),$$

$$\Gamma_{\psi\psi\sigma}(p,q,-p-q) = -\frac{1}{3\sqrt{N}}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\psi\sigma})v - i\frac{1}{6}(\lambda_{\psi\psi} + \delta\lambda_{\psi\psi})\int G_\psi(p+q-k)G_\psi(k)\Gamma_{\psi\psi\sigma}(p+q-k,k,-p-q).$$

Renormalisation and its consistency with bubble summation

Renormalised equation of state and gap equation for M_ψ^2:

$$0 = v\left[m_\sigma^2 + \frac{1}{6}(\lambda_{\sigma\sigma}v^2 + \lambda_{\psi\sigma}T_F(M_\psi^2))\right]$$

$$M_\psi^2 = m_\psi^2 + \frac{1}{6}(\lambda_{\psi\sigma}v^2 + \lambda_{\psi\psi}T_F(M_\psi^2)).$$

Consistent solution of the 6 renormalisation conditions:

$$\delta\lambda_{\psi\psi} + \frac{1}{6}(\lambda_{\psi\psi} + \delta\lambda_{\psi\psi})\lambda_{\psi\psi}T_d = 0, \qquad \frac{\delta\lambda_{\psi\sigma}}{\lambda_{\psi\sigma}} = \frac{\delta\lambda_{\psi\psi}}{\lambda_{\psi\psi}},$$

$$\delta\lambda_{\sigma\sigma} + \frac{1}{6}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})\lambda_{\psi\sigma}T_d = 0$$

$$\delta m_\psi^2 + \frac{1}{6}(\lambda_{\psi\psi} + \delta\lambda_{\psi\psi})\left(\frac{\Lambda^2}{16\pi^2} + m_\psi^2 T_d\right) = 0,$$

$$\delta m_\sigma^2 + \frac{1}{6}(\lambda_{\psi\sigma} + \delta\lambda_{\psi\sigma})\left(\frac{\Lambda^2}{16\pi^2} + m_\psi^2 T_d\right) = 0.$$

Renormalisation of the equation for 3-point coupling and G_σ^{-1}:

$$\Gamma_{\psi\psi\sigma}(p+q) = -\frac{1}{3\sqrt{N}}\lambda_{\psi\sigma}v + \frac{1}{6}\lambda_{\psi\psi}\Gamma_{\psi\psi\sigma}(p+q)I_F(p+q),$$

$$G_\sigma^{-1} = p^2 - \frac{1}{3}v^2\left[\lambda_{\sigma\sigma} + \frac{\lambda_{\psi\sigma}^2}{6-\lambda_{\psi\psi}I_F(p)}I_F(p)\right]$$

$$I(p) = -i\int_k G_\psi(p-k)G_\psi(k) = T_d + I_F(p).$$

Cancellation of infinities in both equations is automatic with the above counterterms

Expression for the vacuum expectation:

$$v^2 = -6\frac{m_\sigma^2 + \frac{\lambda_{\psi\sigma}}{\lambda_{\psi\psi}}(M_\psi^2 - m_\psi^2)}{\lambda_{\sigma\sigma} - \frac{\lambda_{\psi\sigma}^2}{\lambda_{\psi\psi}}} = -6\frac{m_{\mathrm{H}iggs}^2}{\lambda_{\mathrm{H}iggs}}$$

"Effective" Higgs parameters

$$m_{\mathrm{H}iggs}^2 = m_\sigma^2 + \frac{\lambda_{\psi\sigma}}{\lambda_{\psi\psi}}(M_\psi^2 - m_\psi^2) < 0, \qquad \lambda_{\mathrm{H}iggs} = \lambda_{\sigma\sigma} - \frac{\lambda_{\psi\sigma}^2}{\lambda_{\psi\psi}} > 0.$$

CONCLUSIONS

- Transparent non-perturbative renormalisation for 2PI-Hartree approximation to a wide class of scalar field theories with explicit construction of the counterterms

- The construction is applicable also to Dyson-Schwinger equations in the large N limit with ”bubble-sum” improved propagators

Next talk by Gergely Fejős:

- Analysis of the $U(3) \times U(3)$ symmetric meson model in the broken phase

- Solution of the renormalised gap equations plus equations of state with phenomenological mass spectra