Schwinger–Dyson equation: From Minkowski to Euclidean space and back

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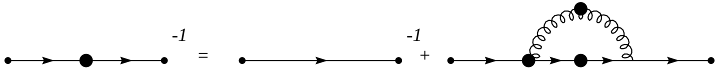
Wick rotation

The (unrenormalized) SDE for the quark–propagator:



$$S^{-1}(p) = \not{p} - m - iC_F g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(k) \Gamma^\nu(k,p) G_{\mu\nu}(p-k)$$

$$S^{-1}(p) = A(-p^2)\not{p} - B(-p^2) = A(-p^2)\left(\not{p} - M(-p^2)\right)$$

Wick rotation

The dressed quark–gluon vertex in the rainbow approximation

$$\Gamma^{\mu}(k,p) \to \gamma^{\mu}$$

The dressed gluon propagator in the Landau gauge

$$g^2 G^{\mu\nu}(k) = \frac{4\pi\alpha_s(-k^2)}{k^2}\left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\right)$$

Wick rotation

Wick rotation $\Rightarrow$ SDE in the Euclidean space

$$
\begin{aligned}
A(x)x &= x + \frac{1}{8\pi^3}\int_0^\infty dy \frac{yA(y)}{A^2(y)y + B^2(y)} k_9(x,y) \\
B(x) &= m + \frac{3}{8\pi^3}\int_0^\infty dy \frac{yB(y)}{A^2(y)y + B^2(y)} k_0(x,y)
\end{aligned}
$$

Wick rotation

$$
\begin{aligned}
k_0(x,y) &= 4\pi C_F \int_0^\pi d\beta \, \sin^2\beta \\
&\times \frac{\alpha_s(x+y-2\sqrt{xy}\cos\beta)}{x+y-2\sqrt{xy}\cos\beta} \\
k_9(x,y) &= 4\pi C_F \int_0^\pi d\beta \, \sin^2\beta \frac{\alpha_s(x+y-2\sqrt{xy}\cos\beta)}{x+y-2\sqrt{xy}\cos\beta} \\
&\times \left(\sqrt{xy}\cos\beta + \frac{2(\sqrt{xy}\cos\beta - y)(x-\sqrt{xy}\cos\beta)}{x+y-2\sqrt{xy}\cos\beta} \right)
\end{aligned}
$$

Wick rotation

Additional approximation to the Schwinger-Dyson equation: $A = 1$

$$M(p^2) = m + 3i \int \frac{d^4k}{(2\pi)^4} \frac{M(k^2)}{k^2 - M^2(k^2)} g(-(k-p)^2)$$

$$g(-k^2) = 4\pi C_F \frac{\alpha_s(-k^2)}{(-k^2)}$$

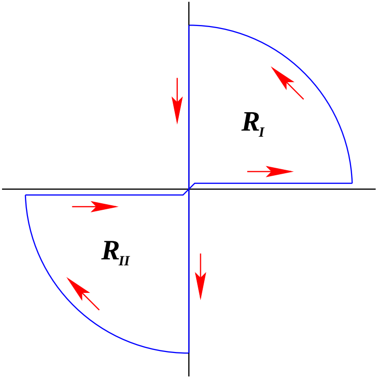
$$f(-k^2) = \frac{3i}{(2\pi)^4} \frac{M(k^2)}{k^2 - M^2(k^2)}$$

$$I(-p^2) = \int d^4k\, f(-k^2) g(-(k-p)^2)$$

Wick rotation

$$M(-p^2) = m + I(-p^2)$$

$$I(-p^2) = \int d^3k \int dk^0 \, f(-(k^0)^2 + |\boldsymbol{k}|^2) g(-(k^0 - p^0)^2 + |\boldsymbol{k} - \boldsymbol{p}|^2)$$



Wick rotation

I define

$$F(k^0) = f(-(k^0)^2 + |\boldsymbol{k}|^2)g(-(k^0 - p^0)^2 + |\boldsymbol{k} - \boldsymbol{p}|^2)$$

The residua theorem gives

$$\begin{aligned} I(x) &= I_E(x) + I_R(x) + I_C(x) \\ &= I_E(x) + 2\pi i \left(\sum_{(k_0)_j \in R_I} \operatorname{Res} F(z) - \sum_{(k_0)_j \in R_{II}} \operatorname{Res} F(z) \right) \end{aligned}$$

- $I_E(x)$ - integral along imaginary k^0 axis
- $I_R(x)$ - poles contribution
- $I_C(x)$ - integral over arcs $= 0$

Wick rotation

SDE in the Euclidean space for $x \in \mathbb{R}$

$$
\begin{aligned}
M(x) &= m + I_E(x) + I_R(x) \\
&= I_R(x) + m + \frac{3}{8\pi^3}\int_0^{+\infty} dy\, \frac{yM(y)}{y + M^2(y)} k_0(x,y) \\
k_0(x,y) &= \int_0^{\pi} d\beta\, \sin^2\beta\, g(x + y - 2\sqrt{xy}\cos\beta)
\end{aligned}
$$

- I have assumed that $I_C(x) = 0$
- $I_R(x)$ is not known a apriori

Models overview

"Gaussian" Ansatz

$$g(Q^2) = C_F \frac{4\pi^2 D}{\omega^2} Q^2 \exp(-\frac{Q^2}{\omega^2})$$

- Jain and Munczek, Phys. Rev. D **48**, 5403 (1993)
- Alkofer, Watson, and Weigel, Phys. Rev. D **65**, 094026 (2002)

Models overview

Bloch's Ansatz (IR part)

$$g(Q^2) = \frac{4\pi C_F}{Q^2} \frac{c_0\, \alpha_0}{c_0 + \left(\frac{Q^2}{\Lambda^2_{\mathrm{QCD}}}\right)^2}$$

- Bloch, J. C. R., Phys. Rev. D **66**, 034032 (2002)

Models overview

Bicudo's Ansatz

$$g(Q^2) = \frac{4}{3}\pi a_1 \left(\frac{1}{\lambda_1^2 + Q^2} - \frac{1}{\lambda_2^2 + Q^2} \right)$$

- P. Bicudo, Phys. Rev. D **69**, 074003 (2004)
- V. Šauli, Czech. J. Phys. **55**, 1205 (2005)

"Gaussian" interaction

Interaction kernel

$$g(Q^2) = C_F \frac{4\pi^2 D}{\omega^2} Q^2 \exp(-\frac{Q^2}{\omega^2})$$

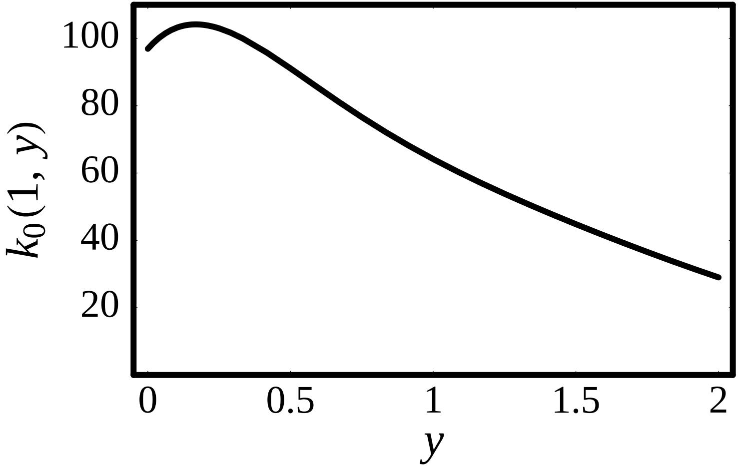
after angular integration

$$k_0(x,y) = \frac{2D\, e^{-\frac{x+y}{\omega^2}} \pi^3 \left((x+y) I_1\left(\frac{2\sqrt{xy}}{\omega^2}\right) - 2\sqrt{xy} I_2\left(\frac{2\sqrt{xy}}{\omega^2}\right) \right)}{\sqrt{xy}}$$

$$k_9(x,y) = 6D\, e^{-\frac{x+y}{\omega^2}} \pi^3 \times \left(-2\sqrt{xy} I_1\left(\frac{2\sqrt{xy}}{\omega^2}\right) + (x+y-2\omega^2) I_2\left(\frac{2\sqrt{xy}}{\omega^2}\right) \right)$$

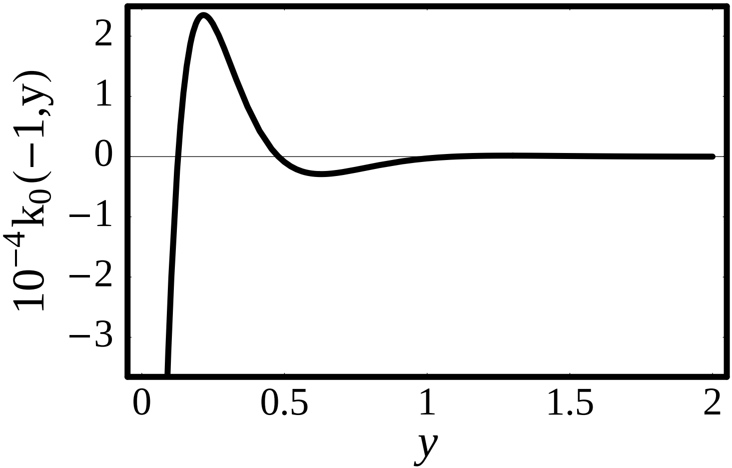
"Gaussian" interaction

Parameters: $D = 16$ and $\omega = 0.5$
Function: $y \mapsto k_0(1, y)$



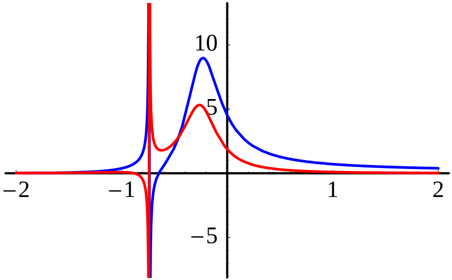
“Gaussian” interaction

Parameters: $D = 16$ and $\omega = 0.5$
Function: $y \mapsto k_0(-1, y)$



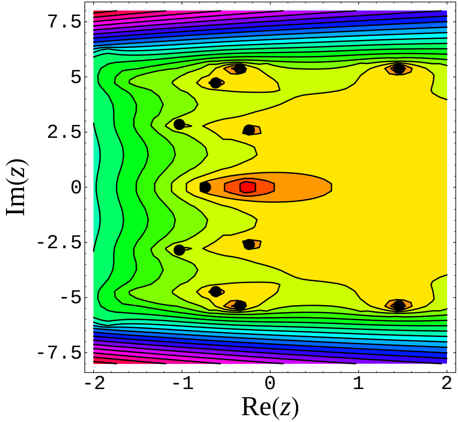
"Gaussian" interaction

Functions $\sigma_A(x)$ (blue) and $\sigma_B(x)$ (red):



- $\sigma_A(x) = \frac{A(x)}{A^2(x)x+B^2(x)}, \quad \sigma_B(x) = \frac{B(x)}{A^2(x)x+B^2(x)}$
- Approximation: no pole contribution, *i.e.*, $I_R(x) = 0$

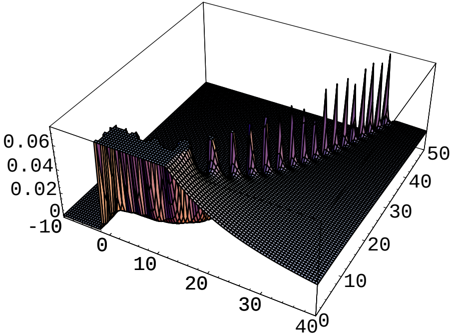
“Gaussian” interaction



Contour plot of the function $z \mapsto \log |A^2(z)z + B^2(z)|$. The black points are solutions of the equation $A^2(z)z + B^2(z) = 0$.

“Gaussian” interaction

3D plot of the function $z \mapsto \left|\frac{1}{A^2(z)z+B^2(z)}\right|$



- There are infinitely many poles in the complex plane
- Functions $z \mapsto A(z)$ and $z \mapsto B(z)$ are analytic

Bloch's Ansatz

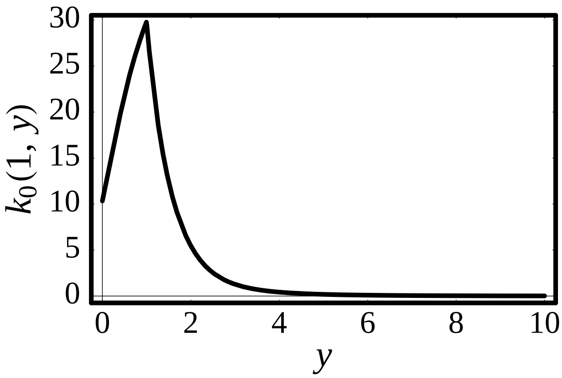
- For $x, y \in \mathbb{R}$, $x > 0$ and $y > 0$

$$
\begin{aligned}
k_0(x,y) &= \frac{i\pi^2 C_F \alpha_0}{2\sqrt{xy}} \Bigg[2\sqrt{-\frac{x}{y} + 2 - \frac{y}{x}} \\
&+ \sqrt{\frac{c_0\Lambda^4_{\mathrm{QCD}} - 2i(x+y)\sqrt{c_0}\Lambda^2_{\mathrm{QCD}} - (x-y)^2}{xy}} \\
&- \sqrt{\frac{c_0\Lambda^4_{\mathrm{QCD}} + 2i(x+y)\sqrt{c_0}\Lambda^2_{\mathrm{QCD}} - (x-y)^2}{xy}} \Bigg]
\end{aligned}
$$

- For $x \in \mathbb{C} \Rightarrow$ more complicated analytic expression

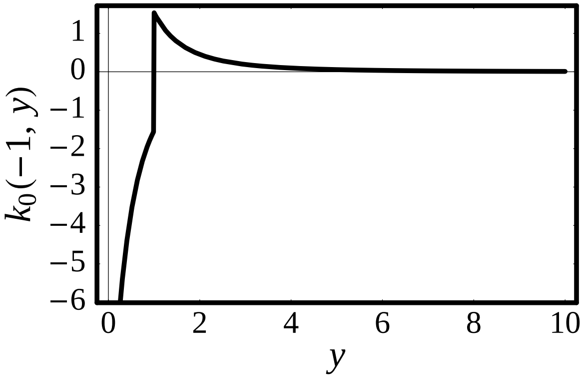
Bloch's Anatz

Function $y \mapsto k_0(1, y)$



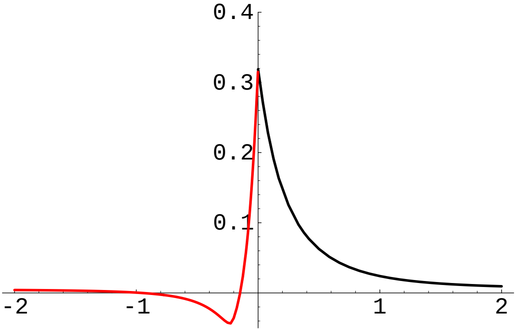
Bloch's Ansatz

Function $y \mapsto k_0(-1, y)$



Bloch's Ansatz

Function $x \mapsto \widetilde{M}(x)$



- approximation $I_R(x) = 0 \;\Rightarrow\; M(x) \to \widetilde{M}(x)$

Bloch's Ansatz

Function $Q^2 \mapsto g(Q^2)$ has the poles for

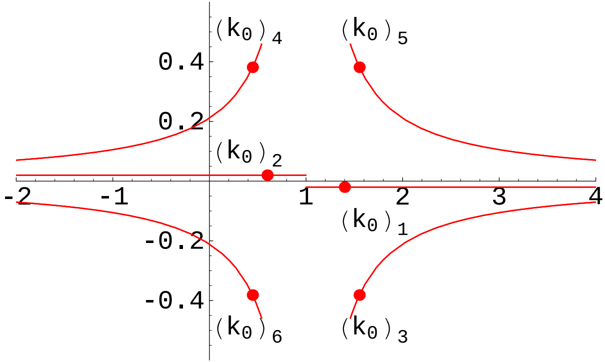
$$Q^2 \in \{i\varepsilon,\, i\sqrt{c_0}\Lambda^2_{\mathrm{QCD}},\, -i\sqrt{c_0}\Lambda^2_{\mathrm{QCD}}\}$$

The poles of the function $k^0 \mapsto g(-(k^0 - p^0)^2 + |\boldsymbol{k} - \boldsymbol{p}|^2)$ are

$$\begin{aligned}
(k^0)_{1,2} &= p^0 \pm |\boldsymbol{k}| \mp i\varepsilon \\
(k^0)_{3,4} &= p^0 \pm \sqrt{|\boldsymbol{k}|^2 - i\sqrt{c_0}\Lambda^2_{\mathrm{QCD}}} \\
(k^0)_{5,6} &= p^0 \pm \sqrt{|\boldsymbol{k}|^2 + i\sqrt{c_0}\Lambda^2_{\mathrm{QCD}}}
\end{aligned}$$

Bloch's Ansatz

Singularities $(k_0)_j$ of the function $k^0 \mapsto g(-(k^0 - p^0)^2 + |\boldsymbol{k}|^2)$ in the complex k^0 plane for $p^0 = 1$ and various values of $|\boldsymbol{k}|$. For example, the red dots emphasize location of the poles for $|\boldsymbol{k}| = 0.4$.



Bloch's Ansatz

What to do?

- $$M^{(0)}(x) = m + \frac{3}{8\pi^3}\int_0^{+\infty} dy\, \frac{yM^{(0)}(y)}{y + (M^{(0)})^2(y)} k_0(x,y)$$
- calculate $I_R^{(0)}(x)$...
- $$M^{(1)}(x) = I_R^{(0)}(x) + m + \frac{3}{8\pi^3}\int_0^{+\infty} dy\, \frac{yM^{(1)}(y)}{y + (M^{(1)})^2(y)} k_0(x,y)$$
- etc...

Problem: $M^{(0)}(z) = \widetilde{M}(x)$ is not analytic: $\oint dz\, \widetilde{M}(z) \neq 0$ generally

Bicudo's Ansatz

Interaction kernel

$$g(Q^2) = 4\pi\alpha \left[\frac{1}{Q^2 + \lambda_1^2} - \frac{1}{Q^2 + \lambda_2^2} \right]$$

after angular integration

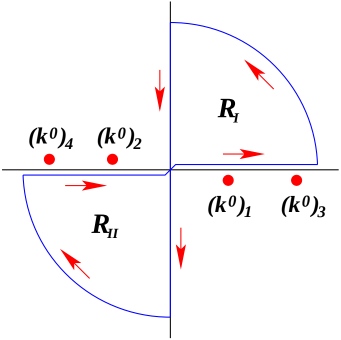
$$k_0(x, y) = \frac{\alpha\pi^2}{xy} \Big[\lambda_1^2 - \lambda_2^2 - \sqrt{(x + y + \lambda_1^2)^2 - 4xy} + \sqrt{(x + y + \lambda_2^2)^2 - 4xy} \Big]$$

- this is for $x, y \in \mathbb{R}$, $x > 0$ and $y > 0$
- For $x \in \mathbb{C} \Rightarrow$ more complicated analytic expression

Bicudo's Ansatz

For spacelike momenta p the poles are

$$
\begin{aligned}
(k^0)_{1,2} &= \pm\sqrt{|\boldsymbol{k}-\boldsymbol{p}|^2+\lambda_1^2} \mp i\varepsilon \\
(k^0)_{3,4} &= \pm\sqrt{|\boldsymbol{k}-\boldsymbol{p}|^2+\lambda_2^2} \mp i\varepsilon
\end{aligned}
$$

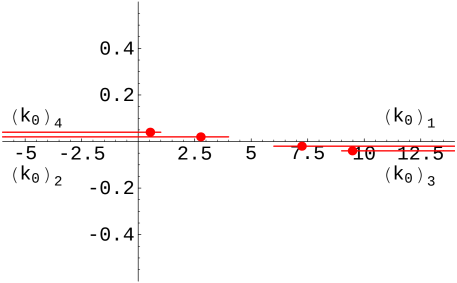


$\Rightarrow \quad I_R(x) = 0$

Bicudo's Ansatz

For timelike momenta p the poles are

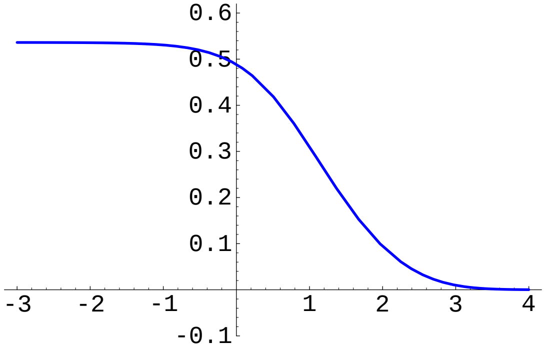
$$(k^0)_{1,2} = p^0 \pm \sqrt{|\boldsymbol{k}|^2 + \lambda_1^2} \mp i\varepsilon$$
$$(k^0)_{3,4} = p^0 \pm \sqrt{|\boldsymbol{k}|^2 + \lambda_2^2} \mp i\varepsilon$$



$\Rightarrow \quad I_R(x) \neq 0$

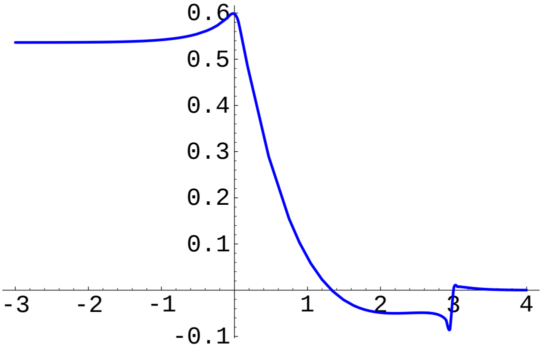
Bicudo's Ansatz

Spacelike region: function $s \mapsto M(10^s)$.



Bicudo's Ansatz

Timelike region: function $s \mapsto \widetilde{M}(-10^s)$.



Bicudo's Ansatz

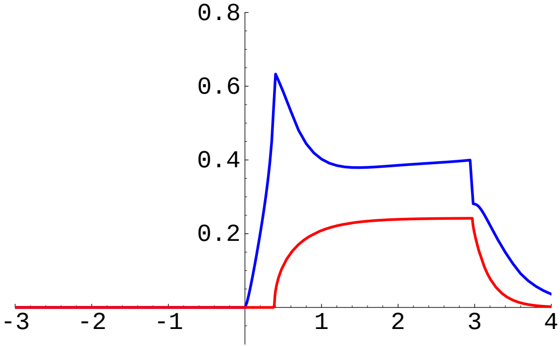
The pole contribution (for $x < 0$)

$$\begin{aligned} & I_R(x) \\ = \; & I_{\mathrm{res},2}(x) + I_{\mathrm{res},4}(x) \\ = \; & i\frac{8}{3}\pi^3 a_1 \int_0^{\max\{0,-x-\lambda_1^2\}} dy \sqrt{\frac{y}{y+\lambda_1^2}} f\left(x - \lambda_1^2 + 2\sqrt{-x}\sqrt{y+\lambda_1^2}\right) \\ + \; & (a_1 \to -a_1, \, \lambda_1 \to \lambda_2) \end{aligned}$$

- $f(x) = -\frac{3i}{(2\pi)^4} \frac{M(x)}{x+M^2(x)}$
- $h(y) = x - \lambda_1^2 + 2\sqrt{-x}\sqrt{y+\lambda_1^2}$ can be negative!
- Can we iterate?

Bicudo's Ansatz

Timelike region: function $s \mapsto I_R(-10^s)$, first iteration



- Blue is the real part and red is the imaginary part

Separable Ansatz

Assuming a Feynman–like gauge

$$g^2 G^{\mu\nu} = g^{\mu\nu} D(-k^2)\ .$$

and a separable Ansatz

$$D(-(p-k)^2) \approx D_0\, f_0(-p^2) f_0(-k^2) - D_1\, (p \cdot k) f_1(-p^2) f_1(-k^2)$$

Separable Ansatz

Solution of SDE:

$$\begin{aligned} A(x) &= 1 + a\, f_1(x) \\ B(x) &= m + b\, f_0(x) \end{aligned}$$

$$\begin{aligned} a &= \frac{1}{2} D_1 C_F \int_0^\infty \frac{x dx}{16\pi^2} \frac{x A(x) f_1(x)}{A^2(x) x + B^2(x)} \\ b &= 4 D_0 C_F \int_0^\infty \frac{x dx}{16\pi^2} \frac{B(x) f_0(x)}{A^2(x) x + B^2(x)} \end{aligned}$$

Separable Ansatz

Form factors:

$$f_0(x) = e^{-x/\Lambda_0^2}$$

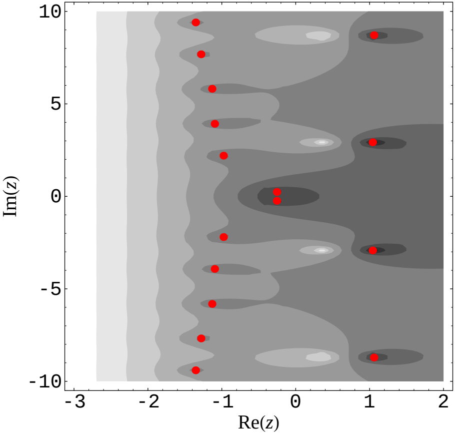
$$f_1(x) = \frac{1 + e^{-x_0/\Lambda_1^2}}{1 + e^{-(x-x_0)/\Lambda_1^2}}$$

Parameters:

$$\Lambda_0 = 0.758 \text{ GeV} \quad \Lambda_2 = 0.960 \text{ GeV} \quad x_0 = 0.6^2 \text{ GeV}^2 \quad m = 5.49 \text{ MeV}$$

$$a = 0.67 \quad b = 0.65 \text{ GeV}^2$$

Separable Ansatz



Model E: Contour plot of the function $z \mapsto \log |A(z)^2 z + B(z)^2|$. The red points are solutions of the equation $A(z)^2 z + B(z)^2 = 0$.

Conclusion

- It is relativly easy to solve SD in the Euclidean space, but it is non-trivial task to extrapolate this solution to the timelike region (or even to the whole complex plane)
- If we need solution of SDE in both the spacelike and timelike region, is it more economic to solve SDE in the Minkowski space?