On the Dyson-Schwinger equations in First Order Formalism

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Outline

1. Motivation
   - Approaches to understand Confinement
   - Infrared behaviour of Coulomb gauge QCD Green Functions
2. Second vs. First Order Formalism
3. Generating Functionals of First Order Formalism
4. Equivalence Relations
5. Feynman Rules
6. Summary and Outlook

Some Selected Approaches to Confinement:

see *e.g.* R.A. and J. Greensite, *Quark Confinement: The Hard Problem of Hadron Physics* , J. Phys. **G34** (Special focus issue on Hadron Physics) (2007) S3.

- chromomagnetic **monopoles**
  't Hooft, diGiacomo, . . .
- center **vortices**
  Greensite, Olejnik, . . .
- **Coulomb** confinement
  Gribov, Zwanziger, . . .
- Landau gauge **Green Functions**
  Smekal, Fischer, . . .
- **AdS**$_5$ / QCD correspondence
  Maldacena, Brodsky, . . .

Theories of Confinement

Properties to be explained:

- String formation
  - Casimir scaling at intermediate distances
  - N-ality at large distances
- Positivity violation of transverse gluons
  - BRST quartet mechanism
  - Oehme–Zimmermann superconvergence relations
- Rôle of Gribov copies
- Conformal IR-YM sector
- DχSB
- $U_A(1)$ anomaly

Note: Not yet understood relations between different approaches, definitely not mutually exclusive.

Interpolating between Landau and Coulomb gauge

IR behaviour of Landau gauge Green functions understood,
see talks by Kai Schwenzer and Markus Huber.

- To uncover relation between Coulomb and Landau gauge:
  Use interpolating gauges $a\partial_0 A_0 + \partial_i A_i = 0$ with $1 \geq a \geq 0$.
- Limit $a \to 0^+$ identical to Coulomb gauge?
  No, additional constraint $\int d^3x A_0 =$const.!
- But:
  propagators in limit $a \to 0^+$ identical to Coulomb gauge ones,
  see lattice calculations by A. Maas *et al.*
- Infrared analysis of Coulomb gauge QCD Green Functions still impossible!!!

D. Zwanziger, Phys. Rev. Lett. **90** (2003) 102001

One-gluon-exchange in Coulomb gauge QCD is confining!

- $D_{00}(\vec{x},t) \propto V_C(|\vec{x}|)\,\delta(t) + \mathrm{non-inst.\ terms}$
- $\lim_{R\to\infty} V_{Wilson}(R) \leq \lim_{R\to\infty} V_C(R)$
- As $V_{Wilson}(R) = \sigma R + \ldots$ one obtains: $V_C(R) = \sigma_c R$ with $\sigma_c \geq \sigma$.
- OVERCONFINING
- lattice calculations: $\sigma_c \approx 3 \times \sigma$, *i.e.* $\sqrt{\sigma_c} \approx 600 \ldots 750$ MeV.

  J. Greensite, S. Olejnik and D. Zwanziger, Phys. Rev. D **69** (2004) 074506 [arXiv:hep-lat/0401003]

  A. Nakamura and T. Saito, Prog. Theor. Phys. **115** (2006) 189 [arXiv:hep-lat/0512042].

Infrared behaviour in the Coulomb gauge

Variational approaches:

- Propagator of transverse gluons vanish in the infrared!

see *e.g.* A. Cucchieri and D. Zwanziger, Phys. Rev. D **65** (2002) 014001
A.P. Szczepaniak, Phys. Rev. D **69** (2004) 074031

$\Rightarrow$ **Transverse gluons confined by positivity violation!**
Cf. gluon propagator in Landau gauge . . .

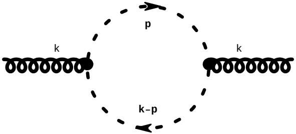
But: Underlying assumptions of variational approaches falsified by lattice calculations!
M. Ilgenfritz, private comm.; Aiko Vogt, diploma thesis, HU Berlin 2007.

Infrared behaviour in the Coulomb gauge

Functional approaches?

Already at perturbative level “catastrophic" divergencies:

Ghost loops in the Coulomb gauge



$$\sim \int \frac{dp_0}{2\pi} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{(\mathbf{k}-\mathbf{p})^2\mathbf{k}^2} \rightarrow \infty$$

- Cancelation in Perturbation Theory?
- Treatment in a Functional Approach like *e.g.* RGs and DSEs**???**

To solve problem change basic formalism!

Second vs. First Order Formalism

Second Order Formalism:

$$\langle 0_{\text{out}}|0_{\text{in}}\rangle = \int \mathcal{D}q \exp\left[\frac{i}{\hbar}\int d^4x\, \mathfrak{L}(q,\nabla q,\dot{q})\right].$$

Symmetries, locality, multiplicative renormalizibility, etc.

are explicitely realized!

First Order Formalism:

$$\langle 0_{\text{out}}|0_{\text{in}}\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left[\frac{i}{\hbar}\int d^4x\, (p\dot{q} - H(q,\nabla q,p))\right].$$

The "momenta" p are independent,

i.e. not related to the fields by the canonical equations.

$\Rightarrow$ mult. renorm. lost!?! locality obscured?!? transf. laws complicated!!!

BUT: ENERGY DIVERGENCE PROBLEM SOLVED!

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Second vs. First Order Formalism

Note:
The renormalizability of the non-abelian gauge theories in Coulomb gauge still unsolved!

Possible constructive way:
Prove equivalence of DSEs in Second (Lagrangian) and First (Hamiltonian) Order formalism.

First task:
Formulate Feynman rules in First Order Formalism to obtain a systematic procedure to express the respective Green functions and Ward (Slavnov-Taylor) identities in the corresponding ones of the Second Order Formalism.*

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*Relation between respective Green functions including their IR behaviour $\Rightarrow$ relation between different Confinement Scenarios.

Generating Functionals of 1stOF

S. Villalba-Chávez, K. Schwenzer and R. Alkofer, to be published

$$\mathcal{Z}^H[J] = \langle 0_{\rm out}|0_{\rm in}\rangle_J = \int \mathcal{D}[q]\mathcal{D}[p] \exp\left\{\frac{i}{\hbar} I\left[q,p,J\right]\right\}.$$

where

$$I = I_0\left[q,p\right] + \int d^4x J(x)\cdot\phi(x)$$

includes

$$\phi(x) \equiv \begin{pmatrix} p_m(x) \\ q_m(x) \end{pmatrix} \text{ and } J(x) \equiv \begin{pmatrix} J_{mp}(x) \\ J_{mq}(x) \end{pmatrix},$$

Note: Action replaced by

$$I_0\left[q,p\right] = \int d^4x \left\{p_m(x)\dot{q}_m(x) - H\left(q,\nabla q,p\right)\right\}.$$

Generating Functionals of 1stOF

Full -> connected:

$$\mathcal{W}^H[J] = -i \ln\left(\mathcal{Z}^H[J]\right),$$

connected -> 1PI:

$$\Gamma^H[\phi] \overset{def}{\equiv} \mathcal{W}^H[J] - \int d^4x J(x) \cdot \bar{\phi}(x).$$

with “classical" fields and momenta s.t.

$$\frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = -J(x).$$

Generating DSEs

for full Green functions:

$$\left.\frac{\delta I_0}{\delta \phi}\right|_{\phi(x)\to\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} \mathcal{Z}[J] = -J(x)\mathcal{Z}[J]$$

for connected Green functions:

$$\left.\frac{\delta I_0}{\delta \phi}\right|_{\phi(x)\to\frac{\delta \mathcal{W}^H}{\delta J(x)}+\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} = -J(x),$$

for 1PI Green functions:

$$\frac{\delta \Gamma^H}{\delta \bar{\phi}(x)} = \left.\frac{\delta I_0}{\delta \phi(x)}\right|_{\phi(x)\to\bar{\phi}[J](x)+\frac{\hbar}{i}\int d^4x'\mathbb{D}[J](x,x')\frac{\delta}{\delta\bar{\Phi}(x')}}$$

Despite formal similarity to respective 2nd order Generating Eqs. these are much more complicated!

If simultaneous infinitesimal transformations

$$\delta\phi[\phi] \equiv \begin{pmatrix} \epsilon\mathfrak{G}_m\,[p,q;x] \\ \\ \epsilon\mathfrak{F}_m\,[p,q;x] \end{pmatrix}$$

leave functional integral “measure" invariant:

$$0 = \int d^4x \frac{\delta\Gamma^H}{\delta\bar{\phi}} \cdot \delta\phi[\phi]\Big|_{\phi\rightarrow\bar{\phi}[J]+\frac{\hbar}{i}\int d^4y\mathbb{D}[J](x,y)\frac{\delta}{\delta\bar{\phi}(y)}} .$$

generates all Ward (Slavnov-Taylor) Identities

Equivalence Relations

n-momenta Green function

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \frac{\int \mathcal{D}[p]\mathcal{D}[q] \frac{\hbar}{i}\frac{\delta}{J_{i_1 p}} \ldots \frac{\hbar}{i}\frac{\delta}{J_{i_n p}} \exp\left[\frac{i}{\hbar}\{I[q,p,J]\}\right]}{\int \mathcal{D}[p]\mathcal{D}[q] \exp\left[\frac{i}{\hbar}\{I[q,p,J]\}\right]}$$

Hamiltonian quadratic in momenta:

$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \frac{\int \mathcal{D}[q] \left\{J_{i_1 p} + \frac{\delta \mathcal{S}}{\delta \dot{q}_{i_1}}\right\} \ldots \left\{J_{i_n p} + \frac{\delta \mathcal{S}}{\delta \dot{q}_{i_n}}\right\} \exp\left[\frac{i}{\hbar}\tilde{\mathcal{S}}\right]}{\int \mathcal{D}[q] \exp\left[\frac{i}{\hbar}\tilde{\mathcal{S}}\right]}.$$

with

$$\tilde{\mathcal{S}} = \mathcal{S} + \frac{1}{2} J_{ip} J_{ip} + J_{ip} \frac{\delta \mathcal{S}}{\delta \dot{q}_i} + J_{iq} q_i$$

where $\mathcal{S}$ is the standard action.
Note: Integrand independent of the p_m.

Equivalence Relations

- DSEs

$$\Delta^{p\dots p}_{i_1\dots i_n} = \left\{ J_{i_1 p} + \frac{\delta \mathcal{S}}{\delta \dot{q}_{i_1}} \right\} \dots \left\{ J_{i_n p} + \frac{\delta \mathcal{S}}{\delta \dot{q}_{i_n}} \right\} \Bigg|_{q_m \to \frac{\delta \mathcal{W}^H}{\delta J_q} + \frac{\hbar}{i} \frac{\delta}{\delta J_q}} .$$

- 1-point function $\langle p_i[J] \rangle$

$$\bar{p}_i[J] = \frac{\delta \mathcal{S}}{\delta \dot{q}_i} \Bigg|_{q \to \frac{\delta \mathcal{W}^H}{\delta J_q} + \frac{\hbar}{i} \frac{\delta}{\delta J_q}} + J_{ip} = \frac{\delta \mathcal{W}^H}{\delta J_{ip}(\mathbf{x}, \tau)} .$$

- Expressed in 1PI Generating Functional

$$\frac{\delta \Gamma^H}{\delta \bar{p}_i} = -\bar{p}_i + \frac{\delta \mathcal{S}}{\delta \dot{q}_i} \Bigg|_{q_m \to \bar{q}_i[J] + \frac{\hbar}{i} \Delta^{qq}_{ij}[J] \frac{\delta}{\delta q_j}} .$$

- provides the identities

$$\Gamma[q] = \mathcal{W}[J_q] - \int d^4x J_{mq}(x) \bar{q}_m(x) \text{ and } \bar{p}_i = \frac{\delta \mathcal{S}}{\delta \dot{q}_i} \Bigg|_{q_m \to \bar{q}_i[J_q] + \frac{\hbar}{i} \Delta^{qq}_{ij}[J_q] \frac{\delta}{\delta q_j}}$$

Equivalence Relations

Performing in

$$0 = \int \mathcal{D}[q] \frac{\delta}{\delta q_m} \int \mathcal{D}[p] \exp\left[\frac{i}{\hbar} I[q,p,J]\right].$$

the $p-$integration provides

$$0 = \int \mathcal{D}[q] \left(\frac{\delta \mathcal{S}}{\delta q_i} + J_{mp} \frac{\delta^2 \mathcal{S}}{\delta q_i \delta \dot{q}_m} + J_{iq} \right) \exp\left[\frac{i}{\hbar} \tilde{\mathcal{S}}[q,J]\right],$$

and thus

$$\frac{\delta \Gamma^H}{\delta \bar{q}_i} = \left(\frac{\delta \mathcal{S}}{\delta q_i} + \left\{ p_m - \frac{\delta \mathcal{S}}{\delta \dot{q}_m} \right\} \frac{\delta^2 \mathcal{S}}{\delta q_i \delta \dot{q}_m} \right) \Bigg|_{q_j \to q^*}$$

where $q^* = \bar{q}_j[J] + \frac{\hbar}{i} \Delta^{qq}_{jk}[J] \frac{\delta}{\delta \bar{q}_k}$.

Equivalence Relations

From the above relations one obtains

$$\frac{\delta \Gamma^L}{\delta \bar{q}_i} = \frac{\delta \mathcal{S}}{\delta q_i}\bigg|_{q_j \to \bar{q}_j[J_q] + \frac{\hbar}{i}\Delta^{qq}_{jk}[J_q]\frac{\delta}{\delta \bar{q}_k}} = -J_{iq}.$$

which is the well-known Generating Functional DSE in the standard Lagrange formalism.

Analogous reformulations:
Slavnov-Taylor identity is reduced to

$$\mathfrak{G}_m[\phi] = \frac{\delta^2 \mathcal{S}}{\delta q_m \delta \dot{q}_n} \mathfrak{F}_m[\phi]$$

Equivalence Relations

Thus we found the required *“translation rules”*:

n-momenta Green functions are calculated from

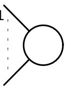
$$\Delta^{p\ldots p}_{i_1\ldots i_n} = \left(\frac{\delta \mathcal{S}}{\delta \dot{q}_{i_1}} \cdots \frac{\delta \mathcal{S}}{\delta \dot{q}_{i_n}} \right) \Bigg|_{q \to \frac{\delta \mathcal{W}}{\delta J_q} + \frac{\hbar}{i} \frac{\delta}{\delta J_q}} ,$$

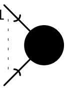
any other m-point Green function with n external legs associated to J_p's can be found by taking $m - n$ derivatives w.r.t. J_{mq} in previous formulas.

Feynman Rules

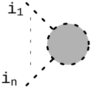
Notations:

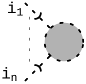
Lagrange formalism

$\triangle^{q\dots q}{}_{,i_1\dots i_n} =$ 

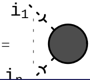
$\Gamma^{L}{}_{,i_1\dots i_n} =$ 

Hamilton formalism, pure momenta Green functions

$\triangle^{p\dots p}{}_{,i_1\dots i_n} =$ 

$\Gamma^{H\,(p\dots p)}{}_{,i_1\dots i_n} =$ 

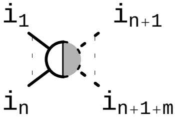
and pure $n-$field proper Green functions

$\Gamma^{H\,(q\dots q)}{}_{,i_1\dots i_n} =$ 

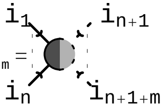
Feynman Rules

Hamilton formalism, mixed $n+m-$proper functions

$\triangle^{q...p}{}_{,i_1...i_{n+m}} =$

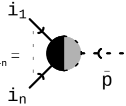


$\Gamma^{H\,(q...p)}{}_{,i_1...i_{n+m}} =$



Additional functions (proper “connectors" coming from the differentiation of $\bar{p}$ with regard the classical fields)

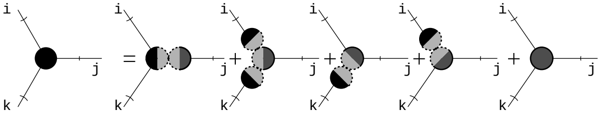
$\bar{p}^{\,(q...q)}{}_{,i_1...i_n} =$



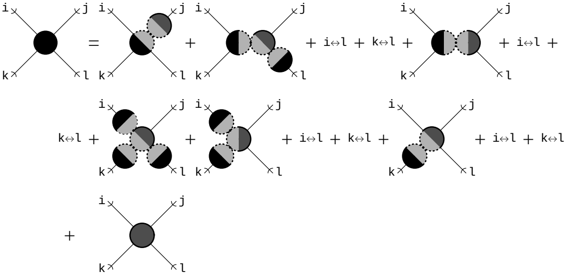
Feynman Rules

Relations for 2-point 1PI functions

3-point 1PI functions



and 4-point 1PI functions



Feynman Rules

Two point proper function

$$\frac{\delta^2 \mathcal{S}}{\delta q_i \delta q_j} \equiv -\frac{\delta^2 I_0}{\delta q_i \delta p_n} \frac{\delta^2 I_0}{\delta p_n \delta q_j} + \frac{\delta^2 I_0}{\delta q_i \delta q_j},$$

Three proper vertex

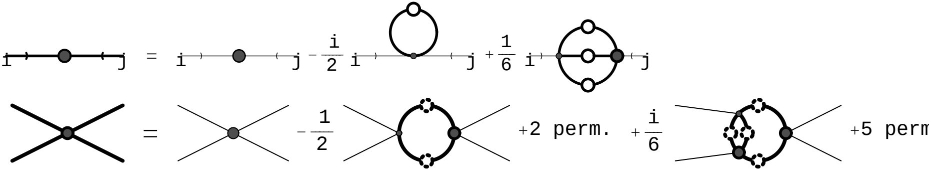
$$\frac{\delta^3 \mathcal{S}}{\delta \bar{q}_i \delta \bar{q}_j \delta \bar{q}_k} = \frac{\delta^3 I_0}{\delta q_i \delta q_k \delta p_n} \frac{\delta^2 I_0}{\delta p_n \delta q_j} + \bar{q}_k \leftrightarrow \bar{q}_j \text{permut.} + \bar{q}_i \leftrightarrow \bar{q}_j \text{permut.}$$

Four-point proper vertex

$$\begin{aligned} \frac{\delta^4 \mathcal{S}}{\delta q_i \delta q_j \delta q_k \delta q_l} &= \frac{\delta^3 I_0}{\delta q_i \delta q_k \delta p_n} \frac{\delta^3 I_0}{\delta p_n \delta q_j \delta q_l} + \bar{q}_i \leftrightarrow \bar{q}_l \text{permut.} \\ &+ \bar{q}_k \leftrightarrow \bar{q}_l \text{permut.} \\ &+ \bar{q}_k \leftrightarrow \bar{q}_l \text{permut.} + \frac{\delta^4 I_0}{\delta q_i \delta q_j \delta q_k \delta q_l}. \end{aligned}$$

Φ^4 theory

Above rules result in



etc.,

and **equivalence is shown straightforwardly.**

Coulomb gauge Yang-Mills theory

Additional vertex

$$\sim g f^{abc} \vec{\Pi}^a \vec{A}^b A_0^c,$$

implies:

- Generation of loops that involve propagators like $\mathfrak{D}_{\Pi A}$, $\mathfrak{D}_{\Pi A_0}$, etc..
- The vacuum expectation value of $\vec{\vec{\Pi}}$ is non-trivial in contrast to scalar field theory and QED.

Summary and Outlook

First Order Functional Approach:

- Formal equivalence of Lagrange and 1st order formalism.
- Explicit relations for 1PI Green functions via Feynman rules and diagrammatic techniques.
- Explicit solution for Φ^4 theory and QED.
- Not unexpected: Yang-Mills theory only implicitely resolvable.

? Renormalization and IR analysis
of Coulomb gauge Yang-Mills theory . . .

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More information . . .

Homepage of the group

Strong Interactions in Continuum Quantum Field Theory:

http://physik.uni-graz.at/itp/sicqft/

Homepage of the FWF-funded Doctoral Program

Hadrons in Vacuum, Nuclei and Stars:

http://physik.uni-graz.at/itp/doktoratskolleg/

C. Gattringer (Lattice), C.B. Lang (Lattice), W. Plessas (Quark Models), W. Schweiger (Exclusive Hadron Reactions), & RA (SICQFT)

Non-equilibrium aspects of Quantum Field Theory:

From cosmology to table-top experiments

Schladming, Styria, Austria, February 23 - March 1, 2008

Peter Arnold (U. Virginia, USA)	Thermalization of Quark-Gluon Plasmas
Wolfgang Cassing (U. Giessen, Germany)	From Kadanoff-Baym dynamics to off-shell parton transport
Thomas Gasenzer (U. Heidelberg, Germany)	Ultracool dynamics far from equilibrium
Hans-Thomas Janka (MPI Garching, Germany)	Stellar Evolution and Death: Physics and Astrophysics
Massimo Pietroni (U. Padova, Italy)	Non-equilibrium in Cosmology
Herbert Schoeller (U. Aachen, Germany)	Functional renormalization group in non-equilibrium
Joachim Stroth (GSI Darmstadt, Germany)	Probing Dense Baryonic Matter with Dileptons

If you wish to apply, please access the web page and complete the registration form as soon as possible, but no later than **February 15, 2008**. More Information about the school can be found on the web page as well.

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